Contract No. Nonr 839(38)

68-29

AD 679652

ON TWO DIMENSIONAL INCOMPRESSIBLE STEADY STATE FLOWS WITH SEPARATION

by

Alexander Pal

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October 1968

POLYTECHNIC INSTITUTE OF BROOKLYN

DEPARTMENT
of
AEROSPACE ENGINEERING
and
APPLIED MECHANICS

PIBAL Report No. 68-29

ON TWO DIMENSIONAL INCOMPRESSIBLE STEADY STATE FLOWS WITH SEPARATION

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Alexander Pal

The research has been conducted under Contract Nonr 839(38) for PROJECT STRATEGIC TECHNOLOGY and was made possible by the support of the Advanced Research Projects Agency under Order No. 529 through the Office of Naval Research.

The subject matter in this report constituted material which satisfied the thesis requirements for the Ph.D. at the Courant Institute of Mathematical Sciences of New York University. The thesis adviser was Professor Joseph B. Keller.

Polytechnic Institute of Brooklyn

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STATE FLOWS WITH SEPARATION

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ABSTRACT

The subjects of this study are two-dimensional incompressible steady state flows which have constant vorticity (ω) in a domain N bounded by a closed streamline and are irrotational in the complementary part P of the flow domain, and such that the streamfunction $\psi(z)$ $(z = x+iy)$ satisfies on the boundary $\gamma = \partial P \cap \partial N$ Bernoully's law

$$(1) \qquad \left(\frac{\partial\psi}{\partial n}\right)_P^2 - \left(\frac{\partial\psi}{\partial n}\right)_N^2 = \lambda = \text{constant} > 0.$$

According to G.K. Batchelor such flows (below called "Batchelor-flows") may be models of laminar flows exhibiting separation phenomena in case of high Reynolds numbers.

Let $f(\zeta)$ $(\zeta = \xi + i\eta)$ denote a regular function in the domain E: $\eta > 0$ ("open flows") or $0 < \eta < \pi$ ("channel-flows") which is $O(|\zeta|)$ as $|\zeta| \to \infty$, and such that on ∂E

(a) $|f'(\zeta)|$ is bounded away from 0, (b) is even in ξ (c) non-increasing if $\xi > 0$, and (d) $f'(\zeta), f''(\zeta)$ satisfy Hölder - conditions and have finite real limits as $\xi \to \infty$.

The existence of Batchelor - type flows in the domain $\Delta = f(E)$, bounded by the streamline(s) $\psi = 0$ and in case of channel flows $\eta = \pi$ is proved by direct methods of variational calculus. In particular, let

$$T[\psi] = \iint_E \left\{ \nabla \left[\psi\left(f(\zeta) \right) \right] \right\}^2 d\xi \, d\eta ,$$

$$L[\psi] = -2 \iint_{\psi(z)<0} \psi(z) \, dx \, dy, \qquad A[\psi] = \iint_{\psi(z)<0} dx \, dy .$$

Then, if $u(z)$ is an arbitrary function for which these functionals are finite, the problems

(III) $\qquad T[u] - \lambda A[u] = \min.$

$\left(\lambda \text{ and } L[\psi] = m > 0 \text{ given parameters} \right)$

and

(IV) $\qquad T[u] - \lambda A[u] - \omega L[u] = \min.$,

$\left(\lambda \text{ and } \omega > 0 \text{ given parameters} \right)$, have solutions which can be considered stream functions of the Batchelor - type, if only λ (and ω in case of problem IV) satisfy certain reasonable inequalities. The region N is then defined by the condition $\psi(z) < 0$, ψ the solution of problems III or IV. The flow is asymptotically uniform at large distances. Further properties of the solution: $\psi\left(f(\zeta) \right)$ is an increasing function of $|\xi|$ and even in ξ. The sets $N \cup \partial N$ and P are simply connected and ∂N contains a finite arc of $\partial\Delta$. For solutions of IV or if $f(\zeta) / \zeta = \text{const.}$, (straight boundaries of Δ) N itself is connected.

Applying the minimum - principle to special variations of ψ near ∂P, it is shown that if z converges in P to the $\psi = 0$ streamline, then

$$\liminf \; |\nabla\psi| \geq \lambda^{\frac{1}{2}} .$$

Consequently γ is rectifiable. By analytic variations of the domain it is shown that ψ and N satisfy an integral equation similar to the one found by P.R. Garabedian and D. C. Spencer in the case of cavitation flows. Properties of the solution, such as boundedness of N and that the matching condition (1) is satisfied along γ almost everywhere, can be deduced from this integral equation.

Solutions of III are never "trivial", i.e. the domain N is never empty. Solutions of IV are non-trivial if λ exceeds a limit dependent on Δ. Such solutions exist unless Δ has straight boundaries.

The set of solutions depends continuously on Δ, λ and ω or m.

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ACKNOWLEDGEMENT

The author wishes to express his gratitude to his Advisor, Professor J. B. Keller, and to Professors P. R. Garabedian, D. Gilbarg and P. D. Lax for stimulating discussions, to Miss P. A. Florence and Mrs. E. W. Thorsen for the preparation of the figure, and to members of the typing pool of the Polytechnic Institute of Brooklyn for careful typing of the manuscript.

This research has been conducted under Contract Nonr 839(38) for Project Strategic Technology, and was made possible by the partial support of the Advanced Research Projects Agency under Order No. 529 through the Office of Naval Research.

Most frequently used notations.

Δ: the physical flow domain in the plane $z = x+iy$, bounded by a streamline β in case of open flows, and the streamlines β, β' in case of channel flows.

E: the halfplane $0 < \eta$ for open flows, the strip $0 < \eta < \pi$ for channel flows of the ζ-plane. ($\zeta = \xi + i\eta$).

$\zeta = g(z)$ the analytic function mapping Δ conformally into the halfplane or strip E. See for details Section 1. 1.

$z = f(\zeta)$ the inverse of $g(z)$.

α, μ: $\alpha = |f'(\infty)|$, $\mu = \inf\{|g'(z)|^2 : z \in \Delta\}$;

Λ, M: $\Lambda = 1/\alpha^2$, $M = 1/|f'(0)|^2$.

If $u(z)$ is a function in Δ, S a subset of Δ, then

$\hat{u}(\zeta) = u(f(\zeta))$,

$\hat{S} = g(S)$.

$u_+(z), u_-(z)$ the positive and negative parts of the function $u(z)$.

$P[u], O[u], N[u]$: for any function $u(z)$ the subset of the interior of the domain of definition of u, where $u > 0, = 0, < 0$ respectively.

$Q[u] = N[u] \cup O[u]$.

γ, Γ: $\gamma = \partial P \cap \partial N$, $\Gamma = \partial P - \beta'$.

(For open flows β' is the empty set.)

$\overline{S}$, Int (S), mes(S), $mes_1(S)$: If S is any set, then $\overline{S}$ is its closure, Int (S) the interior of S, mes(S) the plane measure of S, $mes_1(S)$ its linear measure.

Functionals:	If S is an open set, u, v functions in S, then
$D[u,v\|S]$	$=\iint_S \nabla u \, \nabla v \, dx\, dy,$
$D[u\|S]$	$\equiv D[u, u\|S].$
	If $u(z)$ is a function in Λ, $u(z) = \hat{u}(\zeta)$, then
$T[u\|S]$	$= D[\hat{u}-\eta\|\hat{S}],$
$L[u\|S]$	$= -2\iint_S u(z)\, dx\, dy.$
$T[u], L[u]$:	If S is not specified in these functionals, the integration should be extended over the entire domain of the functions involved.
$A[u]$	$=\iint_{Q[u]} dx\, dy.$
	But:
A_S or $A(S)$	is the plane measure of the set S.
$V[u]$	$=T[u] - \lambda A[u]$
$V'[u]$	$=T[u_+] - m^2 D[u_-]/L[u_-]^2 - \lambda A[u]$
$W[u]$	$=T[u] - \lambda A[u] - \mu L[u].$
λ, m, μ	are specified positive constants.

Function Spaces:	
$\mathcal{T}$	: See Section 1.2
$\mathcal{D}$	: " " 2.1
$\mathcal{T}_P, \mathcal{D}_N$	: " " 2.1
$\mathcal{L}(m)$	: " " 1.3
$\mathcal{H}, \mathcal{H}_+, \mathcal{H}_-$	: " " 2.1
$\mathcal{S}$	: " " 2.1

INTRODUCTION

The problems connected with wakes and cavities appearing in fluid flows have stimulated a great deal of applied and pure mathematical research since Helmholtz described a mathematical model of cavitational flows. An impressive body of mathematical results is available about incompressible irrotational cavity flows. In the plane case, the shape of the cavity can be obtained in special cases by conformal mapping techniques; (cf. e.g., Birkhoff-Zarantanello: Jets, Wakes and Cavities, 1957) and even in the more difficult axially symmetric three dimensional case an explicit solution was constructed by Garabedian (1954). The existence of solutions of flows with cavities behind more general obstacles was shown in the plane and axially symmetric cases by Garabedian and Spencer (1952) and by Garabedian, Lewy, and Schiffer (1952) respectively. Uniqueness of solution is proved by Gilbarg (1952), Serrin (1952). Integral equations are derived by the method of interior variations, and it is shown that the free boundary is an analytic curve. For an extensive bibliography of cavitation flows, see Gilbarg (1960).

The model of flows with cavitation has been widely applied to problems of wakes behind obstacles. This practice may be objected to on the basis that in cavity flow the velocity is constant on the free boundary; however, adjacent fluid particles of the wake region would decelerated the neighboring free stream particles by viscous friction. Admittedly, for unbounded regions there is no assurance that for large Reynolds numbers, R, steady state solutions of the Navier-Stokes equations exist at all, but *if* such a solution exists for R so large that the singular perturbation method of boundary layers is applicable then this solution is probably not well approximated by the cavity model.

The assumption that the flow is irrotational is in general justified by Helmholtz's vortex-theorems, in any region covered by streamlines originating in infinity. Indeed, in the Lagrangian frame of reference the initial condition of zero vorticity for $t = -\infty$ implies no vorticity anywhere along the entire streamline. However, if the flow domain contains a region covered by closed streamlines, then a non-viscous flow becomes indeterminate in this region. In fact, arbitrary constant vorticity may be prescribed along each closed streamline dependent only on the value of the stream-function ψ. This indeterminacy is of course merely a result of the excessive idealization implicit in the Euler-or Lagrange-equations. The indeterminacy disappears if steady state viscous flows are considered for a given set of boundary conditions, such that the Reynolds number R of the flow converges to ∞. Batchelor (1956, 1957) pointed out that in the 2-dimensional case under such circumstances a limit flow may exist, containing regions bounded by closed streamlines, in which the vorticity is constant (eddy regions) and outside which the flow is irrotational.* The eddy regions are separated from each other and from the outside irrotational region by slipstreams (streamlines of velocity discontinuity) which are the limits of boundary layer type velocity distributions. The velocities $\vec{q}_1$, $\vec{q}_2$ on the two sides of a slipstream must satisfy Bernoulli's law, and therefore along the i - th slipstream

$$(1) \quad |\vec{q}_1|^2 - |q_2|^2 = \lambda_i$$

on each slipstreamline. In the simplest case both the eddy region N and the irrotational region P are bounded by the domain-boundary and a single slipstream γ, so that (1) can be written as

$$(2) \quad |\vec{q}_P|^2 - |\vec{q}_N|^2 = \lambda$$

along γ.

In contrast to cavitation-flows, little is known about Batchelor type separated flows. Goldshtik (1962) proved the existence of such flows for $\lambda=0$ by methods of functional analysis in bounded domains. He also showed that if the vorticity ω in the eddy region exceeds a certain value ω_o then at least two solutions exist (other than the trivial solution with no eddy region) and that for $\omega < \omega_o$ no solution other than the trivial exists. Childress (1965) investigated the $\lambda \searrow 0$ case with the asymptotic approximation of slender eddies, and found a simular bifurcation. To my knowledge no rigorous existence-proof exists for the $\lambda > 0$ case, and no special explicit solution for the unknown boundary γ, although both Goldshtik and Childress give numerical

*See also Prandtl (1961).

results under the special assumptions made in their papers.

Mathematical description of the separated flow-problems to be investigated.

Let us introduce the streamfunction ψ (z), (z = x + i y) in the flow-domain Δ, which is simply connected open set bounded by one or two streamlines extending to infinity*. If Δ is bounded by a single streamline β, we will talk about an open flow, in the case of two streamlines (β,β') about a channel flow. (This is not intended as a complete definition of the flow domain; Δ will also be required to satisfy certain additional conditions, which allows Steiner symmetrization of the streamfunction ψ (z). More precise definition will be given in Section 1. 1). It is assumed that Δ is simply covered by streamlines, hence ψ (z) is one-valued on the Riemann surface in which Δ is embedded. The eddy region $N=N[\psi]$ is bounded by β and a slipstreamline γ. We assign the value ψ (z) = 0 to β and γ. In the simplest case γ is the only subset of Δ in which ψ (z) vanishes. Thus, for positive vorticity in N, ψ (z) < 0 in N, and ψ (z) > 0 in $P[\psi] = \Delta - N - \gamma$. ψ (z) should satisfy the equation

$$(3) \qquad \nabla^2 \psi = \omega . s (\psi)$$

---

* It will only be assumed that Δ is locally schlicht. Thus it may cover multiply a plane domain without branchpoints.

in Δ, where $s(\psi)$ is the characteristic function of $N[\psi]$. Further $\psi(z)$ should be continuous on β (and β'), and assume the values of 0 on β (and π on β'). If the set N is bounded (as it will be proved for almost all cases), the asymptotic behavior of $\psi(z)$ is (up to a trivial factor in the case of open flows) determined by the geometry of Δ. This will be discussed in Section 1. 1. On γ in addition to the condition $\psi(z)=0$, we have the matching condition between the normal derivatives of ψ on the two sides of γ:

$$(4) \quad \left(\frac{\partial \psi}{\partial n}\right)_P^2 - \left(\frac{\partial \psi}{\partial n}\right)_N^2 = \lambda$$

obtained from (2).

The purpose of this paper is to prove the existence of a two parameter family of flows in the given domain Δ which satisfies these conditions. (The two parameters are λ and either the vorticity ω or the angular momentum m of the wake region.) As a side result minimum-principles will be derived of which the streamfunctions are solutions. These minimum-principles might prove convenient in the numerical solution of the separated flow-problem. They also offer interesting analogons or extensions of the energy-principles of potential flow theory.

The $\lambda \leq 0$ case will not be treated because it is felt that it has no physical importance. In fact, $\lambda \leq 0$ would correspond to flows, in which $|\vec{q}_P| \leq |\vec{q}_N|$ along γ. In such a case the wake region would continuously loose kinetic energy in the boundary layer along β, which would not be replaced through the boundary layer along the slipstream γ.

It also should be observed that in case of channel flows it is in general unrealistic to assume that a viscous flow with high Reynolds number remains everywhere approximately harmonic in the vicinity of β'. Rather, eddy regions can be expected adjacent to both β and β'. It would be easy to allow eddy regions bounded by $\psi = \pi$ streamlines; but this is for the sake of simplicity not done here. Nevertheless, if β' is straight, we get a realistic flow by reflection of ω and $\psi(z)$ on β'.

This paper will have four parts. Part I contains preliminary results, including the formulation of the minimum-problems. Section 1.1 discusses the flow domain, Section 1.2 introduces a functional analogon to the Dirichlet-integral and the virtual mass, and together with section 1.3 discusses the properties of this functional. Section 1.4 defines related minimum-problems which are formally equivalent to the flow-problem just described. In Sections 1.5 and 1.6 both necessary and sufficient conditions are given under which the functionals appearing in the minimum-problems have lower bounds.

Part II contains the proof that the minimum-problems chosen for investigation have solutions. In particular, 2.1 and 2.2 contain preliminary lemmas on the equicontinuity and lower bound of admissible functions. In 2.3, 2.4, and 2.5 it is shown that if the set $O = O[\psi] \subset \omega$ where ψ vanishes, is a given closed set, then the so obtained "restricted" minimum-problems have solutions, which satisfy (3). Further lemmas needed to clarify the limitations on the vertical and horizontal spread of the eddy region are in Sections 2.6, 2.7, and 2.8. Finally, in 2.9

it is shown that any minimum-sequence of admissible functions contains a subsequence converging to an admissible function, which is thus the solution of the minimum problem.

The unboundedness of the flow domain makes this proof more complex. In the theory of cavitation flows this difficulty is circumvented (see Garabedian-Spencer (1952)) since domains considered there permit Steiner symmetrization relative to <u>both the real and the imaginary axis</u> accompanied by a decrease in the variational functional involved. In the present problem only Steiner symmetrization relative to the imaginary axis will be applied. Therefore additional tools will be needed for the proof of the compactness of the set of competing functions. This is provided in the fundamental lemma 2.8. This lemma essentially states that if a function $\psi(z)$ has its support of finite area A in parallel strip S of unit width, and has a finite Dirichlet integral D, then a unit square subset $S^* \subset S$ exists, such that

$$\left| \iint_{S^*} \psi(z)\,dx\,dy \right| \geq \frac{k}{A^2 D} \left\{ \iint_{S} \psi(z)\,dx\,dy \right\}^3$$

where k is an absolute constant.

In Part III the topological properties of the solution will be investigated. In particular, it will be shown that the domain P is simply connected, the set N is the disjoint union of simply connected open sets (Section 3.4), the set $\bar{N}$ is connected.* This latter result is based on the theorem, interesting

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*I did not succeed in showing in all cases that N itself is connected.

in itself, that in a two-dimensional potential flow around an obstacle B which is free to move without rotation, no equilibrium position of B is possible unless B touches the flow boundary. In particular, it was shown with the aid of the investigation of Schiffer and Szegö (1949) on the properties of Green's function, that the virtual mass as a function of the position of B is a superharmonic function. (Section 3.1). It is further shown by application of the minimum-condition to certain restricted variations of the positive part of the stream function, that the gradient of the latter has in P a positive lower bound (Section 3.5). From this follows easily (Section 3.8) that the boundary γ separating the regions P and N is rectifiable and that the boundary of N contains an arc of β of nonzero length. (Section 3.6). It also follows that the minimum-problems considered have "non-trivial", i.e., not everywhere irrotational solutions in given regions of the (ω, λ) plane (Section 3.7).

In Part IV a variant of the method of interior variations of Garabedian and Spencer (1952) is applied to derive an integral equation for the solution and the curve γ. The method is applied in a halfplane or parallel strip conformal image of Δ, rather than in Δ itself; this results in a simplified calculation and somewhat more explicit form of the integral equation. (Section 4.1). This integral equation is used to show that the matching condition (4) is satisfied almost everywhere on γ (Section 4.2), and that the eddy region is bounded (Section 4.3). The remaining sections contain results on the connectedness of the eddy region.

In the derivation of the matching condition a difficulty not encountered in works on cavitation flows is again the lack of the twofold symmetrization, and that the boundary γ is probably not an analytic curve; in any case, analyticity could not be proved. Although γ is probably smooth, this could not be proved either. Nevertheless, results on the boundary behavior in the theory of the functions analytic in the unit circle, in particular some theorems of Fatou, F. and M. Riesz, and Privaloff helped to overcome this difficulty.

PART I. FORMULATION OF THE MINIMUM PROBLEMS

1.1 The flow domain. Let $\zeta = g(z)$ ($\zeta = \xi + i\eta$, $z = x + iy$) denote a function analytic in Δ which maps Δ in a locally schlicht manner into the domain E, where

E is the $\eta > 0$ halfplane for open flows,

E is the strip $0 < \eta < \pi$ for channel flows.

We may impose the additional conditions that the line β must be mapped into the open real axis, and for channel flows, the line β' onto the open line $\eta = \pi$. Thus $z = \infty$ is mapped into $\zeta = \infty$. It is clearly no restriction of generality to assume that β contains the origin of the z-plane. We may then normalize $g(z)$ by setting $g(0) = 0$, and in the case of open flows, $g'(\infty) = 1$. The inverse of the function g will be denoted as $z = f(\zeta)$. Only such Δ will be considered, for which

(a) $f(\zeta)$ is an even function of ξ and $f(0) = 0$,

(b) $|f'(\zeta)|$ is a constant or a bounded decreasing function of $|\xi|$;

(c) $f'(\zeta)$, $f''(\zeta)$ are bounded in E;

(d) $f'(\zeta)$ and $f''(\zeta)$ have finite limits if $|\zeta| \to \infty$ (along any path), and the former limit is non-zero.

Let us consider now the boundary values

$$\rho(\xi) = \log|f'(\xi)| \qquad (\xi \text{ real})$$

and in case of channel-flows

$$\rho^*(\xi) = \log|f'(\xi + i\pi)| .$$

It will be shown that $\rho(\xi)$ (and $\rho^*(\xi)$) determines $f(\xi)$ uniquely, and in case of open flows find sufficient conditions that the function $\rho(\xi)$ may define an admissible domain-function.

Proposition. The admissible domain-function $f(\zeta)$ is uniquely determined by the boundary values of $|f'(\zeta)|$ on ∂E. If $|f'(\xi + i\eta)|$ is a non-increasing function of $|\xi|$ on the boundary, then it is constant or decreasing in E.

We note first that $f'(\zeta)$ is continuous on ∂E, because $|f''(\zeta)|$ is bounded. Then uniqueness of the harmonic function $\log|f'(\zeta)|$ is a consequence of the Phragmén-Lindelöf theorem and the maximum-principle for both open and channel flows. $\log|f'(\zeta)|$ then determines $\text{ph}\, f'(\zeta)$ up to an additive constant (to be obtained from the symmetry of Δ to the imaginary axis). $f'(\zeta)$ determines then $f(\zeta)$ with the additional condition $f(0) = 0$.

If it is now assumed that $|f'(\xi + i\eta)|$ is a non-increasing function of $|\xi|$ on ∂ E, then the harmonic function $\text{Re}\{f''(\zeta)/f'(\zeta)\}$ has non-positive boundary values on the real axis, is zero on the imaginary axis, and bounded in the right half of E. Therefore by the maximum-principle

$$\text{Re}\{f''(\zeta)/f'(\zeta)\} = \frac{\partial}{\partial \xi}\log|f'(\zeta)| \leq 0$$

in E, where the equality sign holds only if $f'(\zeta)$ = constant. Hence $|f'(\xi + i\eta)|$ is indeed a decreasing or constant function of $|\xi|$ with limit $a > 0$ for $|\xi| \to \infty$.

The functions $f'(\zeta) - a$, $f''(\zeta)$, analytical in E are there bounded by Poisson's theorem, being defined as Poisson's integrals, with bounded boundary values. Since they have zero limits on the boundary for $\xi \to \infty$, by the Phragmén-Lindelöf theorem

$$\lim_{\zeta \to \infty} |f'(\zeta)| = a$$

and

$$\lim_{|\zeta| \to \infty} |f''(\zeta)| = 0.$$

<u>Theorem</u>. Suppose that the function $\rho(\xi)$ is even, non-increasing if $\xi > 0$, and has a limit $\rho(\infty)$ for $\xi \to \infty$. Let the function $\rho(\xi) - \rho(\infty)$ and its first derivative $\rho'(\xi)$ belong to some space $L_p(-\infty, \infty)$ with $p > 1$. Further it is assumed that $\rho(\xi)$, $\rho'(\xi) \in \mathrm{Lip}_\gamma$, i.e. they satisfy Hölder conditions

(1) $|\rho(\xi + h) - \rho(\xi)| < K h^\gamma$,

(2) $|\rho'(\xi + h) - \rho(\xi)| < K h^\gamma$, $(0 < \gamma < 1)$.

Then the relations

(3) $$\log f'(\zeta) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\rho(\theta)}{\theta - \zeta} d\theta$$

(4) $f(0) = (0)$

define an admissible domain function $f(\zeta)$ in E: $\eta > 0$, such that

$$\lim_{|\zeta| \to \infty} f'(\zeta) = e^{\rho(\infty)} = a > 0.$$

In fact, the real part of (3) is a Poisson-integral, and $p(\theta)$ is continuous on the boundary where it assumes the boundary values $\rho(\xi)$. The Hilbert-transform $\sigma(\xi)$ (cf. Titchmarsh, 1937, Chapter V) of $\rho(\xi) - \rho(\infty)$ also belongs by the equivalent of Privaloff's theorem (see Zygmund (1959) § 7.5) to Lip_γ and L_p. Therefore, $\sigma(\xi) \to 0$ as $\xi \to \infty$. Furthermore $f'(\xi)$ has a limit for $\eta \to 0$ and

$$\lim_{\eta\to 0} \operatorname{Im} \log f'(\xi) = \sigma(\xi)$$

or

$$\lim_{\eta\downarrow 0} \log f'(\zeta) = \log f'(\xi) = \rho(\xi) + i\,\sigma(\xi),$$

and

$$\lim_{\xi\to\infty} f'(\xi) = e^{\rho(\infty)} = a. \tag{5}$$

By differentiation of (3) with respect to ξ and subsequent integration by parts

$$\frac{f''(\zeta)}{f'(\zeta)} = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\rho(\theta)-\rho(\infty)}{(\theta-\zeta)^2}\,d\theta = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\rho'(\theta)}{\theta-\zeta}\,d\theta \tag{6}$$

for $\eta \to 0$. Since $\rho'(\theta)$ belongs to L_p and Lip_γ, we find by repetition of the previous argument

$$\lim_{\eta\to 0} \frac{f''(\zeta)}{f'(\zeta)} = \rho'(\xi) + i\,\sigma^*(\xi)$$

where $\sigma^*(\xi)$ is the Hilbert-transform of $\rho'(\xi)$, belongs to L_p and Lip_γ. Thus

$$\lim_{\eta\to 0} f''(\zeta)/f'(\zeta) = f''(\xi)/f'(\xi)$$

(1.1)

also belongs to L_p and Lip_γ. This implies by (5)

$$\lim_{\xi \to \infty} f'(\xi) = 0.$$

Thus it is proved that (c) and (d) are satisfied. The symmetry condition (a) is satisfied because the boundary values are symmetric and they determine $f(\zeta)$ uniquely as a symmetric function; that (b) is satisfied was proved earlier.

A formal analogon of (3) for channel flows is given by

(7) $$\log f'(\zeta) = \frac{1}{\pi i}\int_0^\infty \left\{ \frac{1}{e^{\theta-\zeta}-1} + \frac{1}{e^{\theta+\zeta}-1} \right\} \rho(\theta)\,d\theta$$

$$+ \frac{1}{\pi i}\int_0^\infty \left\{ \frac{1}{e^{\theta-\zeta}+1} + \frac{1}{e^{\theta+\zeta}+1} \right\} \rho^*(\theta)\,d\theta$$

$$= -\frac{1}{\pi i}\int_0^\infty \left\{ \frac{e^{-\theta}-\cosh\zeta}{\cosh\theta-\cosh\zeta}\,\rho(\theta) + \frac{e^{-\theta}+\cosh\zeta}{\cosh\theta+\cosh\zeta}\,\rho^*(\theta) \right\} d\theta$$

where $\rho(\xi)$, $\rho^*(\xi)$ are the boundary values of $\log|f'(\zeta)|$ prescribed on $\eta = 0$, $\eta = \pi$ respectively.

It seems likely that if $\rho(\xi)$, $\rho^*(\xi)$ satisfy suitable differentiability and integrability conditions, then (7) defines an admissible domain-function for E: $0 < \eta < \pi$.

We note that the half plane and the parallel strip are examples of admissible domains. These domains, which can be described by $\Delta = E$, $f(\zeta) = \zeta$, are particularly interesting. Any non-trivial solution III, IV, for such a domain describes a separated flow without apparent "reason" for the separation, since the boundary of Δ is straight*. Such solutions will be called <u>free eddy</u> solutions.

1.2. <u>The extension of the concept of the virtual mass.</u> It was shown by Garabedian and Spencer that the existence problem of Riabouchinsky-flows can be tackled by solving the variational problem of minimum virtual mass. Let $\varkappa$ denote a smooth curve in the upper halfplane joining the points a, b of the real axis. $\varkappa$ and the real axis bound a domain B_1. B_1 and its mirror image in the lower halfplane form a domain B. The virtual mass of B in a flow uniform at large distances is defined as

(1) $$V = \iint_{Z-B} [\nabla(\psi - y)]^2 \, dx \, dy$$

where ψ is harmonic in Z - B and

$$\psi = \operatorname{Im}(z + a/z + \ldots)$$

in the neighborhood of ∞. Then

(2) $$V + A_B = 2\pi a$$

where A_B is the area of B. (cf. Schiffer-Szegö (1949))

We have to introduce a few notations to be used through this treatment :

If S is an open domain, u (z), v (z) functions then

$$D[u, v \mid S] = \iint_S \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) dxdy \equiv 4\iint_S \left| \frac{\partial u}{\partial z}\frac{\partial v}{\partial \bar z}\right| dxdy;$$

---

* However, it is possible to reflect the flow into the lower halfplane, and assume that the eddy is caused e.g. by a flat plate of suitable length and position on the real axis.

$$D[u|S] = D[u, u|S].$$

If S is not specified, then the integrals have to be extended over the domain of definition of the integrand. Thus e.g. if u is defined in Δ,

$$D[u] = 4\iint_{\Delta} \left|\frac{\partial u}{\partial z}\right|^2 dxdy = 4\iint_{E} \left|\frac{\partial \hat{u}}{\partial \zeta}\right|^2 d\xi\, d\eta.$$

We cannot apply the concept of virtual mass for general domains in this form. It can be applied, however, if the integration is extended over E rather than over Δ We define therefore the functional

(3) $$T[u] = \iint_{E} (\nabla_{\zeta} [\hat{u}(\zeta) - \eta])^2 \, d\xi\, d\eta = D[\hat{u} - \eta]$$

where
$$u(z) = \hat{u}(\zeta);\ z = f(\zeta)$$

T is certainly defined for piecewise smooth functions u_o for which $\hat{u}_o(\zeta)=\eta$ outside some circle. If these functions form the space $\mathcal{J}'$, then $T[u]$ is defined further for all elements of $\mathcal{J}$, the closure of $\mathcal{J}'$ in the Dirichlet-metric.

Let $\hat{O}=O[\hat{u}]$ denote the subset of E where $\hat{u} = 0$, and suppose $\hat{O}$ is a measurable set in the plane measure. Then by definition

(4) $$T[u] = D[\hat{u} - \eta | E - \hat{O}] + \text{mes}(\hat{O}),$$

an identity analogous to (2).

Let Σ_r denote a family of open subsets of E for positive values of r such that

$$\Sigma_s \subset \Sigma_r \text{ if } s \leq r$$

and

$$\bigcup_{r=0}^{\infty} \Sigma_r = E \quad .$$

For open flows Σ_r may be the half circle $\{ |\zeta| < r, \ \eta > 0 \}$, for channel flows the rectangle-domain $(-r, r) \times (0, \pi)$.

By Green's identity

$$(5) \qquad T[u] = \lim_{r\to\infty} \left\{ D[\hat{u} \mid \Sigma_r] + \operatorname{Re} \int_{\partial\Sigma_r} (2\hat{u} - \eta)\, d\zeta \right\}.$$

Suppose now that $\hat{u}(\zeta) = \hat{v}(\zeta)$ outside some open bounded set Ω, $(u, v \in \mathfrak{J})$. Then from (5)

$$(6) \qquad T[u] - T[v] = D[\hat{u} \mid \Omega] - D[\hat{v} \mid \Omega]$$

In particular, if $\hat{u} = 0$ for $z \in \overline{\Omega}$ and $\hat{v} = \hat{u}$ on $E - \Omega$, then

$$(7) \qquad T[v] = T[u] + D[\hat{v} \mid \Omega],$$

or by (4)

$$(8) \qquad T[v] = D[\hat{v} - \eta \mid E - \Omega] + D[\hat{v} \mid \Omega] + \text{mes}\, \overline{\Omega}\,.$$

If $v_+(z)$, $v_-(z)$ denote the positive and negative parts of $v(z)$ respectively, and $\hat{P}$ is the set where $\hat{v}(\zeta) > 0$ then (7) and (8) can be given the form

$$T[v] = T[v_+] + D[v_-] = D[\hat{v}_+ - \eta \mid \hat{P}] + D[v_-] + \text{mes}\,(E - \hat{P})$$

This identity remains true even if $\hat{Q} = E - \hat{P}$ is not bounded but has finite area. In fact, we can find a sequence of functions $\{v_n\}$, such that $\hat{v}_n = \hat{v}$ in $\hat{P}$, $\hat{v}_n = 0$ in $\hat{Q} - \Omega_n$ where Ω_n is an open bounded subset of Q and the sequence $\{\hat{v}_n\}$ approximates $\hat{v}$ in the Dirichlet-norm over Q

Then (8) holds for each pair (v_n, Ω_i), hence in the limit for (v, $\hat{\Omega}$).

Suppose that Δ=E is the halfplane $y > 0$ and the function ψ is harmonic over its support Δ - O, and O is a domain bounded away from ∞. Then for open flows

(9) $T[\psi] = \pi a$,

where a is the mass-coefficient defined in (2).

An analogous result holds for channel flows. Suppose that Δ = E, $\psi \in \mathfrak{I}$, and ψ is harmonic in its support $\Delta - \Omega$, and Ω is a set bounded away from β' and from infinity. Then

$\psi = \operatorname{Im}(z + k_{\pm}) + O(e^{-x})$ as $|x| \to \pm\infty$,

and we find easily by Green's identity

(10) $T[\psi] = \pi \operatorname{Re}(k_+ - k_-)$.

1.3 Variation of $T[\psi]$. We will derive a further identity expressing the variation $T[\psi]^* - T[\psi]$ for functions ψ, ψ^* in harmonic over their supports D, D^* respectively. First let E denote the halfplane $\eta > 0$. We assume that D, D^* contain the outside of some half-circle $C_r:\{|\zeta| < r,\ \eta > 0\}$. Then ψ, ψ^* are imaginary parts of analytic functions $\theta(\zeta)$, $\theta^*(\zeta)$ respectively. By reflection $\theta(z)$, $\theta^*(z)$ can be extended into the $\eta < 0$ halfplane. Then $\theta(z)$, $\theta^*(z)$ have first order poles in infinity:

$$(1)\qquad \begin{cases} \theta(\zeta) = \zeta + c + \dfrac{a}{\zeta} + \dots \\ \theta^*(\zeta) = \zeta + c^* + \dfrac{a^*}{\zeta} + \dots \end{cases}$$

(1.3)

(c, c^* real) . By Green's identity

$$(2)\qquad \iint\limits_{D\cap D^*\cap C_r} \nabla \hat{\psi}\cdot\nabla \hat{\psi}^* \, d\xi\, d\eta = \int\limits_{\partial\hat{D}^*} \hat{\psi}\, \frac{\partial \hat{\psi}^*}{\partial n}\, ds + \int\limits_{\partial C_r} \hat{\psi}\, \frac{\partial \hat{\psi}^*}{\partial r}\, ds$$

$$= \int\limits_{\partial\hat{D}^*} \hat{\psi}\, \frac{\partial \hat{\psi}^*}{\partial n}\, ds + \frac{\pi}{2}\, r^2 + \frac{\pi}{2}\,(a^* - a) + O\,(r^{-2}).$$

Interchanging ψ and ψ^* yields

$$(3)\qquad \iint\limits_{D\cap D^*\cap C_r} \nabla \hat{\psi}\, \nabla \hat{\psi}^* \, d\xi\, d\eta = \int\limits_{\partial\hat{D}^*} \hat{\psi}^*\, \frac{\partial \hat{\psi}}{\partial n}\, ds + \frac{\pi}{2}\, r^2 - \frac{\pi}{2}\,(a^* - a) + O\,(r^{-2}).$$

Subtracting the identities (2), (3), we get

$$\int\limits_{\partial\hat{D}} \hat{\psi}^*\, \frac{\partial \hat{\psi}}{\partial n}\, ds - \int\limits_{\partial\hat{D}^*} \hat{\psi}\, \frac{\partial \hat{\psi}^*}{\partial n}\, ds = \pi\,(a^* - a).$$

Therefore, by (1.2.9)

$$T[\psi^*] - T[\psi] = \int\limits_{\partial\hat{D}} \hat{\psi}^*\, \frac{\partial \hat{\psi}}{\partial n}\, ds - \int\limits_{\partial\hat{D}^*} \hat{\psi}\, \frac{\partial \hat{\psi}^*}{\partial n}\, ds,$$

or

$$T[\psi^*] - T[\psi] = -2i \left\{ \int\limits_{\partial\hat{D}} \hat{\psi}^*\, \frac{\partial \hat{\psi}}{\partial \zeta}\, d\zeta - \int\limits_{\partial\hat{D}^*} \hat{\psi}\, \frac{\partial \hat{\psi}^*}{\partial \zeta}\, d\zeta \right\}$$

This identity is clearly invariant to conformal mappings, hence

$$(4)\qquad T[\psi^*] - T[\psi] = -2i \left\{ \int \psi^*\, \frac{\partial \psi}{\partial z}\, dz - \int\limits_{\partial D^*} \psi\, \frac{\partial \psi^*}{\partial z}\, dz \right\}$$

It can be shown similarly that (4) is valid for channel flows as well.

1.4 Minimum-Problems. We introduce the functionals $A[u]$ the area of the domain $N[u]$; ($u = u(z)$ real):

$$A[u] = \iint_{u \leq 0} dxdy,$$

and

$$L[u] = -2\iint_{\Delta} u(z)\, dx\, dy.$$

The quantity $L(u_{-})$ has an interesting physical meaning. A simple computation shows that if $\nabla^2 \psi = \omega$ in $N[\psi]$, then $L[\psi_{-}]$ is the angular momentum of the flow in $N[\psi]$.

We will also use the notation

$$L[u \mid S] = -2\iint_{S} u(z)\, dx\, dy$$

where S is an open domain.

Definition. The classes of all continuous functions $u(z)$, such that $L[u_{-}] = m$ will be denoted as $\mathcal{L}(m)$, and the class for which $A[u] = b$ as $\mathcal{B}(b)$.

We will also use the notations

$$\mathcal{I}\mathcal{B}(b)\ \mathcal{L}(m) = \mathcal{A}_{I}(b, m) \equiv \mathcal{A}_{I} \quad *$$

$$\mathcal{I}\mathcal{B}(b) = \mathcal{A}_{II}(b) \equiv \mathcal{A}_{II} \quad *$$

$$\mathcal{I}\mathcal{L}(m) = \mathcal{A}_{III}(m) \equiv \mathcal{A}_{III} \quad *$$

Minimum problem I. Find a function $\psi \in \mathcal{A}_{I}(b, m)$, such that for any $u \in \mathcal{A}_{I}(b, m)$,

$$T[\psi] \leq T[u].$$

---

*Here the notation
$$U\,V = U \cap V$$
was used. (U, V are arbitrary classes of functions)

(1.4)

A weaker version of this problem only requires

$$\delta\, T\,[u] = 0 \text{ if } u = \psi, \qquad (u \in \mathcal{A}_I)$$

or more precisely,

$$\frac{\partial}{\partial \theta} \Big\{ T\,[\psi + \theta v] \Big\}_{\theta = 0} = 0$$

for all $v = v\,(z)$, for which $\psi + \theta v \in \mathcal{A}_I$ for all sufficiently small values of $|\theta|$. (Variational problem I, to be distinguished in the future from minimum-problem I.)

If we introduce suitable Lagrange multipliers, other equivalent variational problems arise:

II. $\delta\, U\,[u] \equiv \delta \{ T\,[u] - \omega L\,[u_-] \} = 0$

if $u \in \mathcal{J}\mathcal{B}(b)$;

III. $\delta\, V\,[u] \equiv \{ \delta\, T\,[u] - \lambda\, A\,[u] \} = 0$

if $u \in \mathcal{J}\mathcal{L}(m)$;

IV. $\delta\, W\,[u] \equiv \delta \{ T[u] - \omega L\,[u] - \lambda\, A\,[u] \} = 0$

if $u \in \mathcal{J}$.

To each variational problem we can formulate a corresponding minimum-problem. We cannot expect however these minimum-problems to be equivalent to each other.

There is an equivalent formulation of minimum-problem III. Given any function $u\,(z) \in \mathcal{J}$, we set

$$u_1\,(z) = \begin{cases} u\,(z) & \text{if } u\,(z) \geq 0 \\ m\,\dfrac{u\,(z)}{L\,[u_-]} & \text{if } u\,(z) < 0. \end{cases}$$

(1.4)

Then $u_1(z) \in \mathcal{J}\mathcal{L}(m)$. Thus III is equivalent to III' :

Functional minimized:

$$V'[u] = T[u_1] - \lambda A[u]$$
$$= T[u_+] + m^2 D[u_-]/L[u_-]^3 - \lambda A[u],$$

where in the second equation the identity (1.2.7) was used.

Constraints: None.

Given Data: m, λ

Competing functions: $\mathcal{J}$,

An equivalent version of I can be formulated similarly. (See Table I.)

Min. Problem	Functional Minimized	Constraints	Parameters Given	Space of Admissible Functions
I	$T[u]$	$L[u] = m$, $A[u] = b$	m, b	$\mathcal{A}_I(m, b) \equiv \mathcal{J}\mathcal{L}(m)\mathcal{B}(b)$
I'	$T'[u] = T[u_+] + m^2 D[u_-] / L[u_-]^2$	$A[u] = b$	m, b	$\mathcal{A}_I'(b) \equiv \mathcal{A}_{II}(b) = \mathcal{J}\mathcal{B}(b)$
II	$U[u] = T[u] - \omega L[u_-]$	$A[u] = b$	ω, b	$\mathcal{A}_{II}(b)$
III	$V[u] = T[u] - \lambda A[u]$	$L[u] = m$	m, λ	$\mathcal{A}_{III}(m) = \mathcal{J}\mathcal{L}(m)$
III'	$V'[u] = T'[u] - \lambda A[u]$	--	m, λ	$\mathcal{A}_{III}' = \mathcal{A}_{IV} = \mathcal{J}$
IV	$W[u] = T[u] - \lambda A[u] - \omega L[u]$	--	ω, λ	$\mathcal{A}_{IV} = \mathcal{J}$

Table I.

1.5 <u>Lower Bounds of the functionals T, U, V, W.</u> First step in solving any of the minimum-problems I-IV is of course to establish the existence of a lower bound of the functionals, the minimum of which is asked for.

For all $u \in \mathcal{J}$, $T[u] > 0$ by definition. To find a lower bound for $U[u]$, $u \in \mathcal{A}_{II}(\omega, b)$, we first have to prove a

<u>Faber-Krahn-type inequality.</u> Let u (z) be a function such that u has a finite Dirichlet-integral and its support S has a finite area A_S. Then

(1) $\quad \alpha[u] = A_S^2 \; D[u] / L[u]^2 \geq 2\pi,$

where $\quad L[u] = -2 \iint_S u(z)\, dx\, dy.$

Equality holds if and only if S is a circle and u satisfies there

$\nabla^2 u = \text{const.}$ in S,

u continuous everywhere, i.e.

$u = \kappa \max(1 - |z - z_0|^2, 0)$

where κ is a real constant, z_0 fixed.

For, it is clear that Schwarz symmetrization of the function u (z) around z_0 leaves $L[\psi]$ and A_S invariant. On the other hand it is known that (see Pólya -Szegö 1951) Schwarz-symmetrization decreases the Dirichlet integral of ψ, hence the result.

By the inequality (1)

$$U[u] > D[u_-] - \omega L[u_-]$$

$$\geq \frac{2\pi L[u_-]^2}{A[u]^2} - \omega L[u_-] \geq -\frac{\omega}{8\pi} A[u]^2 = -\frac{\omega}{8\pi} b^2.$$

(1.5)

Hence $U[u]$ has a lower bound for $u \in \mathfrak{A}_{II}$ dependent only on b and ω. This also implies that $V[u]$ and $W[u]$ always have lower bounds dependent on (λ, m) or (λ, ω) only if the area of Δ is finite.

We will show now that W is unbounded from below and hence IV has no solution for open flows if $f'(\zeta) \to 1$ as $|\zeta| \to \infty$. Let us define for $r > 0$ the functions $u_r(z) = \hat{u}_r(\zeta)$. We set $\zeta_1 = \zeta - 2r$,

$$\hat{u}_r(\zeta) = \begin{cases} \operatorname{Im}[\zeta - 2r + 4r^2/(\zeta - 2r)] & \text{in } \hat{P} = \{\zeta : |\zeta - 2r| > 2r,\ \eta > 0\} \\ \frac{\omega}{4}[|\zeta - (2+i)r|^2 - r^2] & \text{in } \hat{N} = \{\zeta : |\zeta - (2+i)r| < r\} \\ 0 & \text{in } \hat{O} = E - \hat{P} - \hat{N}. \end{cases}$$

Clearly $u_r(z) \in \mathfrak{J}$. In fact by (1.2.7)

$$T[u_r] = 4\pi r^2 + D[\hat{u}_r | \hat{N}] = (\pi/16)\,\omega^2 r^2 + 4\pi r^2.$$

Further

$$L[(u_r)_-] \geq M_r \iint_{\hat{N}} \hat{u}_r(\zeta)\, d\xi\, d\eta = (\pi/8) M_r\, \omega r^4$$

where

$$M_r = \inf\{|f'(\zeta)|^2 : |\zeta - (2+i)r| < r\}$$

and similarly

$$A[u_r] \leq \pi M_r r^2.$$

Hence

$$W[u_r] \leq (\pi/16)\,\omega^2 r^2 (1 - 2M_r) + (4 - \lambda)\pi r^2.$$

With $r \to \infty$, $M_r \to 1$, hence $W[u_r] \to -\infty$.

We will investigate now the existence of lower bounds for $V[u]$, $W[u]$ in general. As before, we will use the notations

$$a = \lim_{|\zeta| \to \infty} f'|(\zeta)|$$

and

$$\Lambda = 1/a^2.$$

In addition we set

$$X = a^2 \lambda = \lambda/\Lambda,$$

$$Y = \frac{1}{12}\left(\frac{\omega}{\pi\Lambda}\right)^2$$

We will consider only $\lambda \geq 0$, i.e., $X \geq 0$.

Definition. We will say that a minimum problem III or IV is wellposed, if for any set $S \subset \mathcal{A}_{III}(m)$ or $\mathcal{A}_{IV}$ in which $V[u]$, $W[u]$ respectively are bounded, $A[u]$ is bounded uniformly for $u \in S$.

Proposition. If III is wellposed, then $T[\psi]$ has an upper bound for $\psi \in S$ dependent on $\sup_S V[\psi]$ alone. Obvious.

If IV is wellposed, then $T[\psi]$ has an upper bound for $\psi \in S$, dependent on $\sup_S W[\psi]$ alone. In fact, if $A<B$ and $W[\psi]<\Omega$ for $\psi \in S$, $(A \equiv A[\psi])$

$$D[\psi_-] - \omega L[\psi_-] \leq U[\psi] = W[\psi] + \lambda A \leq \Omega + B,$$

where B is dependent on Ω alone since problem IV was assumed wellposed. Therefore by (1.5.1)

$$2\pi (L[\psi_-]/B)^2 - \omega L[\psi_-] \leq \Omega + B.$$

This inequality implies that

$$L[\psi_-] \leq \frac{B^2}{4\pi}\left[\omega + \{\omega^2 + 8\pi(\Omega + B)/B^2\}^{1/2}\right],$$

(1.6)

hence from $T[\psi] = V[\psi] + \omega L[\psi-]$,

$$T[\psi] \leq \Omega + B + \frac{\omega}{4\pi} B^2 \left[\omega + \left\{\omega^2 + 8\pi \frac{\Omega + B}{B^2}\right\}^{1/2}\right]$$

1.6 <u>Theorem on the existence of lower bounds for $V[u]$, $W[u]$.</u>

(a) For both open and channel flows, it is necessary for the existence of a lower bound for $V[u]$ over $\mathcal{A}_{III}(m)$ that $X \leq 1$, and it is sufficient that $X < 1$.

(b) For open flows $W[u]$ has no lower bound over $\mathcal{A}_{IV} = \mathcal{J}$ (at least if $\alpha > 0$ as postulated)

(c) For channel flows, it is necessary for the existence of a lower bound of $W[u]$ over $\mathcal{A}_{IV}$ that

(1) $\quad X \leq 1$ and $Y \leq \Phi(X)$,

and it is sufficient that

(2) $\quad X < 1$ and $Y < \Phi(X)$,

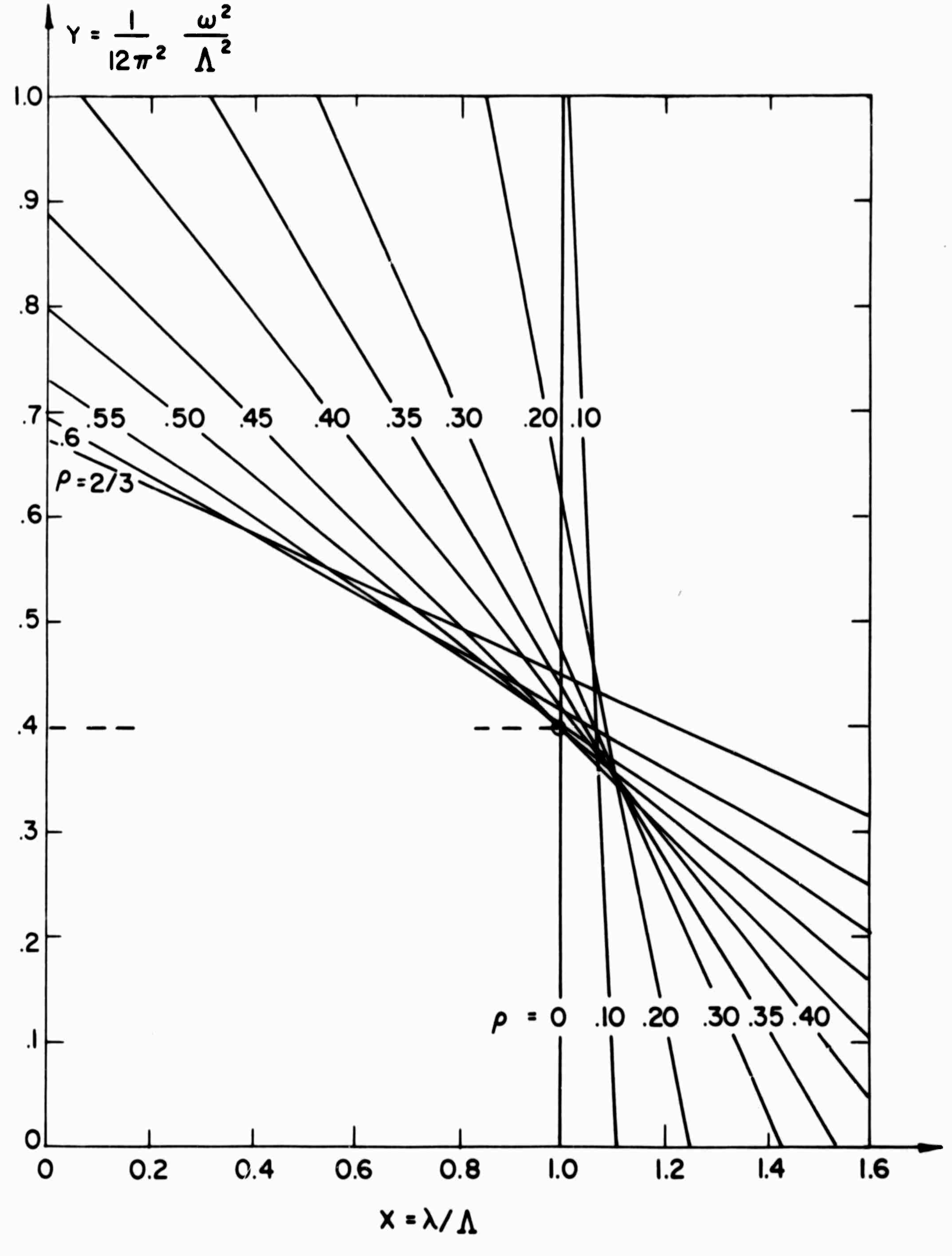
where $Y = \Phi(X)$ represents the envelope of the family of straight lines

(3) $\quad X + \rho^2 Y = \frac{1}{1-\rho}$, $\frac{1}{2} < \rho < \frac{2}{3}$.

(See the figure)

(d) If $X < 1$, then problem III is wellposed. Similarly, if in (c) the strict inequalities (2) hold, then IV is wellposed.

<u>Proof.</u> We will prove (a),(c),(d) only for channel flows, since the proof for open flows is entirely similar. (b) was already proven, hence it will be sufficient to examine $V[\psi]$ and $W[\psi]$ only for channel flows. Let us first establish sufficient conditions for the existence of lower bounds.



$Y = \frac{1}{12\pi^2} \frac{\omega^2}{\Lambda^2}$
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
.55
.50
.45
.40
.35
.30
.20
.10
.6
$\rho = 2/3$
$\rho = 0$
.10
.20
.30
.35
.40
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
$X = \lambda / \Lambda$

(1.6)

(i) We introduce the functionals

$$D_1[v] = \int_{-\infty}^{\infty} d\xi \int_0^{\pi} \left(\frac{\partial v}{\partial \eta}\right)^2 d\eta < D[v],$$

$$T_1[u] = D_1[u-\eta] < T[u]$$

and the function $H = H(\xi)$ the measure of the support of $u_-(\xi + i\eta)$ for $\eta > 0$, fixed ξ. We find by application of a trivial modification of (1.2.6), (1.2.7).

(4) $T_1[u_+] = \int_{\hat{P}} \left(\frac{\partial \hat{u}}{\partial \eta} - 1\right)^2 d\xi d\eta + A_{\hat{Q}}$,

(5) $T_1[u] = T_1[u_+] + \int_{\hat{N}} \left(\frac{\partial \hat{u}}{\partial \eta}\right)^2 d\xi\, d\eta = T_1[u_+] + D_1[u_-]$.

Let us reflect $\hat{u}$ on the real axis, and apply subsequent Steiner-sy..a-metrization relative to the real axis. It is known that the integral

(6) $\int_{-\pi}^{+\pi} \left[\left(\frac{\partial \hat{u}_+}{\partial \eta} - \mathrm{sign}(\eta \hat{u}_+)\right)\right]^2 d\eta = \int_{-\pi}^{+\pi} \left[\left(\frac{\partial \hat{u}_+(\xi, \eta)}{\partial \eta}\right)\right]^2 d\eta - 2[\pi + H(\xi)]$

is not increased by Steiner-symmetrization. On the other hand, we note that $L[\hat{u}_-]$ and $A[\hat{u}]$ remain unchanged by it. We may assume therefore that the set $Q[\hat{u}]$ and $\hat{u}_+(\xi, \eta)$ are already symmetrised, hence $\hat{u}_+(\xi, \eta)$ is a non-decreasing function of $|\eta|$. For such $\hat{u}$ by Schwarz's inequality

$$\int_0^{\pi} \left(\frac{\partial \hat{u}_+}{\partial \eta}\right)^2 d\eta \geq \frac{\pi^2}{\pi - H}$$

(1.6)

Introducing this estimate into (6) we find

$$\int_0^{\pi} \left(\frac{\partial \hat{u}_+}{\partial \eta} - \operatorname{sign} \hat{u}_+\right)^2 d\eta \geq \frac{H(\xi)^2}{\pi - H(\xi)} .$$

Therefore by (4) and (5)

$$\text{(7)} \quad T_1[u] \geq \int_{-\infty}^{\infty} \frac{H(\xi)^2}{\pi - H(\xi)} d\xi + A_{\hat{Q}} = \int_{-\infty}^{\infty} \frac{H^2}{\pi - H} d\xi + \int_{-\infty}^{\infty} H d\xi$$

$$= \pi \int_{-\infty}^{\infty} \frac{H(\xi)}{\pi - H(\xi)} d\xi .$$

Given any function $v(\eta) > 0$ of support S of measure $\leq H$, the minimum of

$$\Omega[v] = \int_S \left(\frac{\partial v}{\partial \eta}\right)^2 d\eta \Big/ \left(\int_S v \, d\eta\right)^2$$

is achieved if S is the interval $(0, H)$ and $v = \eta(H-\eta) \equiv v_0$. Thus

$$\Omega[v] \geq \Omega[v_0] = 12/H^3 .$$

Applying this result to the estimation of $D_1[u_-]$, we find

$$\text{(8)} \quad D_1[u_-] \geq 12 \int_{-\infty}^{\infty} \left\{ \int_0^{\pi} u_-(\xi + i\eta) \, d\eta \right\}^2 \frac{d\xi}{H^3} = 3 \int_{-\infty}^{\infty} \frac{K(\xi)^2}{H(\xi)^3} d\xi ,$$

where

$$K = K(\xi) = -2 \int_0^{\pi} u_-(\xi + i\eta) \, d\eta .$$

(1.6)

Further

(9) $$L[u_-] = -2\iint_E |f'(\zeta)|^2 u_-(\zeta)\, d\xi\, d\eta \le \int_{-\infty}^{\infty} p(\xi) K(\xi)\, d\xi,$$

where

$$p = p(\xi) = \sup\{|f'(\xi + i\eta)|^2 : 0 < \eta < \pi\}.$$

Combination of (8) and (9) yields

$$D_1[u_-] - \omega L[u_-] \ge \int_{-\infty}^{\infty} \left(\frac{3}{H^3} K^2 - p\omega K\right) d\xi.$$

The smallest value of the right hand integrand for all values of K is $-(1/12)\,\omega^2 p^2 H^3$, hence

(10) $$D_1[u_-] - \omega L[u_-] \ge -\frac{\omega^2}{12} \int_{-\infty}^{\infty} p(\xi)^2 H(\xi)^3\, d\xi.$$

Finally

(11) $$A_Q = \iint_{\hat{Q}} f'(\zeta)^2\, d\xi\, d\eta \le \int_{-\infty}^{\infty} p(\xi) H(\xi)\, d\xi.$$

Combining (7) and (11), (7), (10) and (11) respectively, we find

(12) $$V_1[u] \equiv T_1[u] - \lambda A_Q \ge \int_{-\infty}^{\infty} \left(\frac{\pi}{\pi - H} - \lambda p\right) H\, d\xi;$$

(13) $$W_1[u] \equiv V_1[u] - \omega L[u_-] \ge \int_{-\infty}^{\infty} \left(\frac{\pi}{\pi - H} - \lambda p - \frac{\omega}{12} p^2 H^2\right) H\, d\xi.$$

If (2) is valid then we can select an $X' > X$ and in case of IV a $Y' > Y$ such that (2) is still satisfied even in X', Y' is substituted for X, Y.

(1.6)

Since

$$\lim_{\xi \to \infty} p(\xi) = a^2 = 1/\Lambda ,$$

we can choose ξ_o so large that for $\xi > \xi_o$

$$\lambda \, p(\xi)^2 < X' \quad \text{(III and IV)}$$

and

$$\frac{\pi}{12} \mu \, p(\xi)^2 < Y' \quad \text{(IV)} .$$

But then, with $H/\pi = \rho = \rho(\xi)$, by (2) and (3)

$$\int_{\xi_o}^{\infty} \left(\frac{\pi}{\pi - H} - \lambda p^2 \right) H d\xi \geq \pi \int_{\xi_o}^{\infty} \left(\frac{1}{1 - \rho} - X' \right) \rho \, d\xi > 0 ,$$

(problem III), and

$$\int_{\xi_o}^{\infty} \left(\frac{\pi}{\pi - H} - \lambda p - \frac{\mu}{12} p^2 H^2 \right) H d\xi \geq \pi \int_{\xi_o}^{\infty} \left(\frac{1}{1 - \rho} - X' - \rho^2 Y' \right) \rho \, d\xi > 0 ,$$

(problem IV). Therefore by (12) and (13)

$$V[u] \geq V_1[u] \geq 2 \int_0^{\xi_o} \left(\frac{\pi}{\pi - H} - \lambda p \right) H d\xi$$

$$\geq - 2\pi \xi_o \Lambda u^2 ,$$

$$W[u] \geq W_1[u] \geq 2 \int_0^{\xi_o} \left[\frac{\pi}{\pi - H} - \lambda p - \frac{\mu}{12} p^2 H^2 \right] H d\xi$$

$$\geq - 2\pi \xi_o \left(\Lambda + \frac{\mu}{12} \pi^2 \right) \mu^2 ,$$

where we took into consideration

$$p(\xi) \leq \sup_E f' \left| (\zeta) \right|^2 = \mu^2 .$$

(1.6)

(ii) We show that if $X > 1$, or if $Y > \Phi(X)$, then $W[u]$ has no lower bound. Let h denote any number in $(0, \pi)$ and ℓ any positive number. Then we introduce the functions

$$v(\eta) = \begin{cases} -(\omega/2\Lambda)\,\eta\,(h-\eta) & \text{if } 0<\eta<h, \\ \pi\,\dfrac{\eta - h}{\pi - h} & \text{if } h<\eta<\pi, \end{cases}$$

$$w(\xi, \ell) = \begin{cases} 1 \text{ if } \ell < \xi < 2\ell \\ 0 \text{ if } \xi < \ell - 1, \text{ or } \xi > 2\ell + 1 \\ \text{linear in } (\ell - 1, \ell) \text{ and in } (2\ell, 2\ell + 1)\,; \end{cases}$$

$$u(\xi + i\eta, \ell) = v(\eta) w(\xi, \ell) + [1 - w(\xi, \ell)]\eta\,,$$

and the rectangle domains

$$R_\ell = (\ell, 2\ell) \times (0, h)\,, \quad S_\ell = (\ell, 2\ell) \times (h, \pi),$$

further the non-negative number

$$M_\ell = \inf\{|f'(\zeta)|^2 : \zeta \in R_\ell\}.$$

If Q denotes again the domain where $\hat{u}(\zeta) = u(z) \le 0$, A_Q its area, then obviously

(14) $$h < A_Q < h + 2\pi.$$

A simple calculation shows that constants C_1, C_2 independent of ℓ and h exist such that

(15) $$T[u_+] - A_Q < D[v - \eta \mid S_\ell] + C_1 = \frac{h^2}{\pi - h}\,\ell + C_1$$

(16) $$D[u_-] < D[v \mid R_\ell] + C_2 = \frac{\omega^2}{12\Lambda^2}\,h^3 \ell + C_2$$

(1.6)

and that

(17) $$L[u_-] > -2M_\ell \iint_{R_\ell} v(\eta)\, d\xi\, d\eta = \frac{M_\ell\, 2}{6} h^3 \ell ;$$

(18) $$A[u] = \iint_{\hat{N}} |f'(\zeta)|^2 d\xi\, d\eta > M_\ell h .$$

Combining Eqs. (15), (16), (17), (18) we find

(19) $$\begin{cases} V[u] = T[u_+] + D[u_-] - \lambda A[u] \\ < \left(\dfrac{\omega^2}{12\Lambda^2} h^3 + \dfrac{\pi h}{\pi - h} - \lambda M_\ell h\right)\ell + C_3 \end{cases}$$

(20) $$\begin{cases} W[u] = V[u] - \omega L[u_-] \\ < \left(\dfrac{\omega^2}{12\Lambda^2} h^3 + \dfrac{\pi h}{\pi - h} - \dfrac{\omega^2}{6\Lambda} M_\ell h^3 - \lambda M_\ell h\right)\ell + C_3 , \end{cases}$$

(C_3 independent of h and ℓ). Since

$$\lim_{|\xi| \to \infty} |f'(\zeta)|^2 = a^2 = 1/\Lambda ,$$

uniformly in $0 \leq \eta \leq \pi$, to any number $\epsilon > 0$ we can find an ℓ such that

$$M_\ell > \frac{1}{\Lambda} + \epsilon .$$

We substitute this inequality into (19), and get

(21) $$V[u] < \left[\frac{\pi h}{\pi - h} + \frac{\omega^2}{12\Lambda^2} h^3 - \frac{\lambda}{\Lambda} h + \left(\frac{\omega^2}{6}\pi^3 + \lambda\pi\right)\epsilon\right]\ell + C_3$$

$$= h\left[\frac{1}{1 - h/\pi} - X + \left(\frac{h}{\pi}\right)^2 Y\right]\ell + C_3 + C_4\, \epsilon\, \ell ,$$

and similarly

(22) $$W[u] < h\left[\frac{1}{1-h/\pi} - X - \left(\frac{h}{\pi}\right)^2 Y\right]\ell + C_3 + C_4\,\varepsilon\,\ell.$$

Clearly $V[u]$ has no lower bound if $X > 1$, since we can choose then a positive h, such that in (21) the last expression in brackets becomes negative, then fixing h, for $\ell \to \infty$, $V[u] \to -\infty$. Similarly, if $X > 1$ or $Y > \Phi(X)$, then there exists a value of h such that in (22) the expression in brackets is negative. Then clearly $W[u] \to -\infty$ as $\ell \to +\infty$.

This completes the proof of parts (a), (b), (c) of the theorem.

Statement (d) is a consequence of (11), (12), (13). In fact, we can write (11) in the form

(11') $$A_Q \leq \alpha^2 \int_{-\infty}^{\infty} H(\xi)\,d\xi$$

By hypothesis numbers $X' > X$, $Y' > Y$ can be found which still satisfy (2), i.e., such that

(23/III) $$1 - X' = a > 0$$

and

(23/IV) $$\inf_{0 \leq \rho < 1} \left\{\frac{1}{1-\rho} - X' - \rho^2 Y'\right\} = b > 0$$

respectively. Taking into consideration that $p(\xi) \to \alpha^2$ as $\xi \to \infty$, we find from (12) and (13) that a finite C exists such that

$$V[v] \geq \int_{-\infty}^{+\infty} H(\xi)\left[\frac{1}{1 - H(\xi)/\pi} - X'\right] d\xi - C,$$

$$W[u] \geq \int_{-\infty}^{+\infty} H(\xi)\left[\frac{1}{1 - H(\xi)/\pi} - X'\left(\frac{H(\xi)}{\pi}\right)^2 Y'\right] d\xi - C.$$

Introducing inequalities (23) and (11') into these inequalities, we find

$$A_Q \leq \{V[u] + C\}\, \alpha^2/a \qquad \text{(III)}$$

$$A_Q \leq \{W[u] + C\}\, \alpha^2/b \qquad \text{(IV)}$$

from which the statement (d) follows.

Remark. From the contents of this chapter it is clear that these minimum-problems are not equivalent. However, any solution of a well-posed IV is also a solution of I, II, III, with suitably chosen parameters ($m = 0$, $b = 0$ must be included). Any solution of a well-posed III or a problem II is also a solution of a problem I.

We will further consider only III and IV, since attempts to show that N has a rectifiable boundary were unsuccessful for I, II.

When we speak about III or IV, we will always mean a well-posed minimum-problem III or IV.

Well-posed problems may be characterized equivalently by replacing the condition of boundedness of $A[u]$ by a condition that $T[u]$ be bounded for all function $u(z)$ for which $V[u]$ and $W[u]$ respectively are bounded. In fact, by (1.2.4)

$$T[u] = T[\hat{u}] \geq A[\hat{u}] \quad ,$$

and since it was assumed that $|f'(\zeta)|$ has the upper bound μ, this implies

$$\text{(24)} \quad A[u] = \iint\limits_{Q[\hat{u}]} |f'(\zeta)|^2\, d\xi\, d\eta \leq \mu^2 \iint\limits_{Q[\hat{u}]} d\xi\, d\eta \leq \mu^2\, T[u].$$

PART II

CONVERGENCE OF THE MINIMIZING SEQUENCE

2.1 Definitions

$\mathcal{H}_+$ is the family of non-negative continuous functions harmonic in their supports. $\mathcal{H}_-$ is the family of non-positive continuous functions u which satisfy

$$\nabla^2 u = \omega$$

in their supports with any constant $\omega > 0$.

$\mathcal{H}$ is the set of all continuous functions $u(z)$, such that

$u_+(z) \equiv \max(u(z), 0) \in \mathcal{H}_+$,

$u_-(z) \equiv \min(u(z), 0) \in \mathcal{H}_-$

$\mathcal{J}(j)$ is the set of all functions $u(z)$ in $\mathcal{J}$, such that $T[u] < j$.

$\mathcal{J}_P$ is the set of all functions in $\mathcal{J}$ having subsets of P as their supports.

$\mathcal{D}$ is the Dirichlet-norm closure of the set of all functions which have finite Dirichlet-integrals.

$\mathcal{D}_N$ is the set of all functions in $\mathcal{D}$ having subsets of N as their supports.

$\mathcal{S}$: the set of functions $\psi(z) = \hat{\psi}(\xi + i\eta)$ even in ξ, and decreasing functions of $|\xi|$ in E.

A relationship between D[u], L[u] and ω. Suppose that $u \in \mathcal{H}_- \mathcal{D}$. Then by Green's theorem applied to u in the domain $N_\varepsilon = \{z : u(z) < -\varepsilon\}$ $(\varepsilon > 0)$

(2.2)

$$D\left[u \mid N_\epsilon\right] = \iint_{N_\epsilon} (\nabla u)^2 \, dx\, dy = \int_{\partial N_\epsilon} u \frac{\partial u}{\partial n} ds - \iint_{N_\epsilon} u \nabla^2 u \, dx\, dy$$

$$= - \int_{N_\epsilon} \frac{\partial u}{\partial n} ds - \omega \iint_{N_\epsilon} u \, dx\, dy = - \epsilon \iint_{N_\epsilon} \nabla^2 u \, dx\, dy + (\omega/2) L\left[u \mid N_\epsilon\right]$$

$$= - \epsilon\, \omega\, A(N_\epsilon) + (\omega/2) L\left[u \mid N_\epsilon\right] .$$

Letting $\epsilon \to 0$, we find that for any $u \in \mathcal{H}_- \mathcal{D}_-$

(1) $$2\, D\,[u] = \omega\, L\,[u] .$$

2.2 Elements of $\mathcal{J}(j)\mathcal{H}\mathcal{L}(m)$ and of $\mathcal{J}(j)\mathcal{H}\mathcal{A}_{IV}(\omega)$ are equicontinuous.

Proof. It is sufficient to show that there are positive increasing functions $\Phi_P(d)$, $\Phi_N(d)$ such that $\Phi_P(d) \to 0$, $\Phi_N(d) \to 0$, and dependent only on the parameters j, m or j, ω besides d, for which

(1) $$|u(z) - u(t)| \leq \Phi_P(|z - t|)$$

if $z \in P[u]$, $t \in \overline{P[u]}$;

(2) $$|u(z) - u(t)| \leq \Phi_N(|z - t|)$$

if $z \in N[u]$, $t \in \overline{N[u]}$.

It is sufficient to consider only $|z - t| = d < 1$.

Case a. By a well known lemma used in the theory of the Dirichlet-problem, to any $\mu > 1$, there is a circle $C = \{z : |z - t| < \rho\}$, $d \leq \rho \leq \mu d$, such that on any two points p, q of ∂C

$$(3) \qquad |u(p) - u(q)| \le \left(\frac{2\pi D[u|C]}{\log \mu}\right)^{\frac{1}{2}} .$$

(3) remains valid even if u is replaced by $u_+(z)$, or equivalently, if C is replaced in (3) by $C' = C \cap P[u]$. We estimate now the righthand Dirichlet integral in (3).

$$D[u-y \mid C'] = D[u \mid C'] - 2D[u, y \mid C'] + D[y \mid C'].$$

Here

$$D[u, y \mid C'] = \int_{\partial C'} u\, dx .$$

However, if $\zeta \in \hat{P} = g(P)$,

$$\hat{u}(\zeta) \le \eta$$

because of the maximum principle of harmonic functions. Thus

$$D[u \mid C'] = D[\hat{u} \mid \hat{C}'] = D[\ddot{u} - \eta \mid \hat{C}'] + A(C')$$

$$\le T[u] + \rho^2 \pi \max_{\Delta} |g'(z)|^2 \le j + \mu^2 \Lambda d^2 \pi .$$

We substitute this and $\mu = d^{-\frac{1}{2}}$ into (3) :

$$|u(p) - u(q)| \le \left(\frac{4\pi j}{\log(1/d)}\right)^{\frac{1}{2}} + \Lambda \pi d \equiv \Phi_P(d)$$

for any $p, q \in \partial C'$. By the maximum principle the same estimate remains valid if p is replaced by any z in C and q by t.

Case b. With the same meaning of C as before, C can be replaced in (3) by $C'' = C \cap N[\psi]$. From (3) we deduce, considering also $D[u \mid C''] \le T[u] \le j$, (a consequence of Eq. (1.2.7)),

(4) $$|u^*(p) - u^*(q)| \leq \left(\frac{2\pi j}{\log \mu}\right)^{\frac{1}{2}} + \frac{\omega}{2}\mu^2 d^2 ,$$

where $u^*(z) = u(z) - \frac{\omega}{4}|z-t|^2$. The function $u^*(z)$ is harmonic in C'', therefore by the maximum principle (4) is valid if p, q are replaced by t, z respectively, z an arbitrary point in C''. Hence, returning to the function $u(z)$, we find

(5) $$|u(z) - u(t)| \leq \left(\frac{2\pi j}{\log \mu}\right)^{\frac{1}{2}} + \frac{\omega}{2}\mu^2 d^2$$

We substitute now $\mu = d^{-\frac{1}{2}}$, ($d < 1$ was assumed) into (5), and find

(6) $$|u(z) - u(t)| \leq \left(\frac{4\pi j}{\log (1/d)}\right)^{\frac{1}{2}} + \frac{\omega}{2} d \equiv \Phi_N(d) .$$

If $u \in \mathcal{J}(j)\mathcal{H}\mathcal{L}(m)$, then by (2.1.1) ω can be replaced in (6) by $2j/m$.

Corollary. Elements of the spaces $\mathcal{J}(j)\mathcal{H}\mathcal{L}(m)$ and of $\mathcal{J}(j)\mathcal{H}\,\mathcal{A}_{IV}(\omega)$ have a common lower bound.

2.3. The restricted minimum-problems.

Suppose that the sets P, O, N are specified, such that a $\psi_0 \in \mathcal{J}$ exists for which $P[\psi_0] = P$, $N[\psi_0] = N$, $O[\psi_0] = O$. P and N are determined uniquely by O.

The set of all functions $u \in \mathcal{J}$ for which

$u > 0$ in P,

$u = 0$ in O,

and $u < 0$ in N,

form a subset $\mathcal{J}^o \subset \mathcal{J}$. We define the restricted minimum-problems III' and IV:

(2.4)

Find $\psi \in \mathcal{J}^o$ such that for any $u \in \mathcal{J}^o$

$$V'[u] \geq V'[\psi] \; , \qquad (III')$$

$$W[u] \geq W[\psi] \qquad (IV)$$

We define the class $\mathcal{J}_P \subset \mathcal{J}^o$ of functions $v \geq 0$ of support P. Let $\mathcal{D}_N$ denote the closure of the space of continuous differentiable non-positive functions with support N. The restricted minimum-problem splits into the following two problems:

A) Outer minimum-problem

Find $\psi_+(z) \in \mathcal{J}_P$ such that for any $u \in \mathcal{J}_P$,

$$T[u] \geq T[\psi_+] \; ,$$

B) Inner minimum-problem

Find $\psi_-(z) \in \mathcal{D}_N$, such that for any $v \in \mathcal{D}_N$,

$$\frac{D[v]}{L[v]^2} \geq \frac{D[\psi_-]}{L[\psi_-]^2} \qquad (III')$$

or

$$D[v] - \omega L[v] \geq D[\psi_-] - \omega L[\psi_-] \qquad (IV)$$

Because of (1.2.7) it is clear that the function $\psi(z) = \psi_+(z) + \psi_-(z)$ is a solution of the restricted minimum-problem.

2.4 The solution of the outer minimum problem

We first consider the case that $\hat{P} = g(P)$ is bounded by a finite number of Jordan-arcs. Let $\psi^*(\zeta)$ be the solution of the Dirichlet-problem for

$\hat{P}$ with the boundary value assignment

$$\psi^*(\zeta) = -\eta$$

on $\partial\hat{P}$, ψ^* bounded in $\hat{P}$. We set then $\hat{\psi}_+(\zeta) = \psi^*(\zeta) + \eta$. If $u(z)$ is an arbitrary element of $\mathcal{J}_P$, then by elementary identities

$$D[u^*] = D[\psi_+^*] + D[\psi_+^* - u^*],$$

or

$$(1) \qquad T[u] = T[\psi_+] + D[u - \psi],$$

Thus ψ_+ is the solution of the outer minimum-problem, as shown by Polya (1947).

The more general case is that $\hat{P}$ is any open domain in E, the class $\mathcal{J}_P$ is not empty. We approximate P by increasing connected open domains P each of which is bounded by Jordan-curves:

$$P_1 \subset P_2 \subset \ldots \subset P_k \subset P_{k+1} \subset \ldots \subset P = \bigcup_k P_k.$$

By the maximum-principle the corresponding harmonic function $\hat{\psi}_n(z)$ are monotonic domain-functions. For fixed z, $y > 0$,

$$\hat{\psi}_1(\zeta) \leq \hat{\psi}_2(\zeta) \leq \ldots \leq \hat{\psi}(\zeta) \leq \eta,$$

hence by Harnack's second theorem they converge to a function $\hat{\psi}_+(\zeta)$ harmonic in $\hat{P}$. The functions $\hat{\psi}_n(z)$ are equicontinuous in $\overline{P}$ because of lemma 2.2, hence $\psi_+(z)$ is also continuous in $\overline{P}$, and vanishes on ∂P. Let u(z) denote any function in $\mathcal{J}_P$. Then u(z) can be approximated in the Dirichlet-norm by a sequence $u_n(z) \in \mathcal{J}_P$. By Eq. (1) then

$$(2) \qquad T[u_n] = T[\psi_n] + D[u_n - \psi_n].$$

By the definition of the functions u_n,

$$\lim_{n\to\infty} D[u_n - u] = 0.$$

From the last equation by the triangle inequality of Dirichlet-integrals

(3) $$\lim_{n\to\infty} T[u_n] = T[u] .$$

Furthermore by the lower semi-continuity of the Dirichlet-integral

(5) $$\underline{\lim}_{n\to\infty} T[\psi_n] \geq T[\psi], \quad \underline{\lim}_{n\to\infty} D[u_n - \psi_n] \geq D[u - \psi] .$$

Combining (3), (4), (5)

$$T[u] \geq T[\psi] + D[u - \psi] > T[\psi]$$

if $u \not\equiv \psi$.

We observe first that by the maximum principle

$$\psi_n(z) \geq 0 \quad \text{if} \quad z \in \overline{P}_k$$

and therefore the same inequality applies to ψ in $\overline{P[\psi]}$. By the maximum-principle therefore

(4) $$\psi(z) > 0 \quad \text{if} \quad z \in P .$$

Second, if ψ is the solution of the outer problem, then by the uniqueness of the solution of Dirichlet's problem $T[\psi]$ depends on the domain P alone, and may be considered a domain functional, say,

$$T[\psi] = \inf\left\{T[u]:\ u \in \mathcal{J}_P\right\} \equiv \tau[P] .$$

$\tau[P]$ is a decreasing domain functional. For, if $P_1 \subset P$ is an admissible outer domain, i.e., it is open and the class $\mathcal{J}_{P_1}$ is not empty, then $\mathcal{J}_{P_1} \subset \mathcal{J}_P$, implying that

$$\tau[P_1] = \inf\left\{T[u] : u \in \mathcal{J}_{P_1}\right\} \geq \inf\left\{T[u] : u \in \mathcal{J}_P\right\} \equiv \tau[P] .$$

2.5 Solution of the restricted inner minimum-problems.

First assume again that N is a bounded open set bounded by a finite number of Jordan-curves. We will prove the following lemma:

(i) For such N there is a function $\Psi(z)$ which satisfies

(1) $$\nabla^2 \Psi = 1 \qquad (z \in N)$$

and

(2) Ψ is continuous and vanishes on ∂N.

(ii) Let $\psi_{III} = (m/L[\Psi])\Psi(z)$, $\psi_{IV} = \omega\,\Psi(z)$

Then for any $u \in \mathcal{D}_N\ \mathcal{L}(m)$, $v \in \mathcal{D}_N$,

(3) $$D[u] = D[\psi_{III}] + D[u - \psi_{III}],$$

(4) $$D[v] - \omega L[v] = D[\psi_{IV}] - \omega L[\psi_{IV}] + D[v - \psi_{IV}],$$

hence ψ_{III}, ψ_{IV} are solutions of the inner problems III, IV respectively.

Proof. Let z_o denote a fixed complex number, $\Psi^*(z)$ the solution of the Dirichlet problem for N with the boundary value assignment

$$\Psi^*(z) = -(1/4)\,|z - z_o|^2 \text{ if } z \in \partial N.$$

We set

$$\Psi(z) = \Psi^*(z) + (1/4)\,|z - z_o|^2 \text{ in } N.$$

Then indeed

(5) $$\nabla^2 \Psi = 1$$

and Ψ is continuous and vanishes on ∂N. For any function $w(z) \in \mathcal{D}_N$, and real α,

(6) $$D[w] = D[\alpha\Psi] + D[w - \alpha\Psi] + 2D[\alpha\Psi,\ w - \alpha\Psi]$$

$$(2.5)$$

Here by Green's identity and by (5)

$$2D[\alpha\Psi,\ w-\alpha\Psi] = 2\alpha\int_{\partial N}(w-\alpha\Psi)\frac{\partial\Psi}{\partial n}\,ds - 2\alpha\iint_N (w-\alpha\Psi)\nabla^2\Psi\,dx\,dy$$

$$= \alpha(L[w] - L[\alpha\Psi]).$$

If we set now $w = u$, $\alpha = m\ /\ L[\Psi]$, $\alpha\Psi = \psi_{III}$, then because of $L[u] = m$,

$$2\,D[\psi_{III}, u - \psi_{III}] = 0,$$

hence substituting the last identity into (6) yields (3). On the other hand, the substitutions $w = v$, $\alpha = \omega$, $\alpha\Psi = \psi_{IV}$ lead similarly to (4).

We show now that the statement of the lemma is true even if N is any open set of finite area. Then we approximate N by increasing bounded open sets bounded by finite numbers of Jordan-curves:

$$N_1 \subset N_2 \subset \ldots \subset N = \bigcup_{\nu=1}^{\infty} N_\nu .$$

Let z_o be any point in N. The distance d_k of z_o from N_k is less than the distance d from N. Hence, and because of the inequality (2.2.2), the functions Ψ_ν are uniformly bounded. The functions

$$\Psi^*_\nu(z) = \Psi_\nu(z) - \frac{1}{4}|z - z_o|^2$$

are harmonic. They form a decreasing sequence for any $z \in N$. The proof of this property is the same as for the analogous outer problem. The functions Ψ^*_k have a common lower bound. Therefore by Harnack's second theorem they have a pointwise limit $\Psi^*(z)$ harmonic in N. The function

$$\Psi(z) = \Psi^*(z) + |z-z_o|^2/4$$

will therefore satisfy (1). The functions $\Psi_\nu(z)$ are equicontinuous in $\overline{N}$ cf. lemma 2.2, thus Ψ is continuous there and vanishes on ∂N. Furthermore

(7) $$\ell_n = L[\Psi_n] \to L[\Psi] = \ell ,$$

and because of the lower semicontinuity of the Dirichlet-integral,

(8) $$D[\Psi] \leq \underline{\lim}_{n\to\infty} D[\Psi_n] .$$

Thus, setting $\psi_\nu(z) = (m/\ell_\nu)\, \Psi_\nu(z)$

(9) $$\psi_\nu(z) \to (m/\ell)\, \Psi(z) \equiv \psi_{III}(z) ,$$

uniformly in N, ψ_{III} is continuous and vanishes on ∂N, and from (7), (8)

$$L[\psi_{III}] = m, \quad D[\psi_{III}] \leq \underline{\lim}_{\nu\to\infty} D[\psi_\nu] .$$

The functions $\psi_\nu(z) = (m/\Lambda_\nu)\, \Psi_\nu(z)$ satisfy (3) if we replace there N by N_ν. Thus

(10) $$D[u_\nu] = D[\psi_\nu] + D[u_\nu - \psi_\nu] .$$

Suppose that $u \in \mathcal{D}_N\, \mathcal{L}(m) = \mathcal{A}_N(m)$. Then we can approximate u by a sequence $u_\nu \in \mathcal{A}_{N_\nu}(m)$ in the Dirichlet-norm so that

(11) $$D[u_\nu - u \mid N] \to 0, \quad D[u_\nu] \to D[u] .$$

Hence indeed by (9), (10) and (11)

$$D[\psi_{III}] \leq D[u] - D[u - \psi_{III}] < D[u] ,$$

unless $u \equiv \psi_{III}$.

The corresponding proof for problem IV differs only in trivial details.

We observe that the maximum - principle associated with the equation (1) implies that for all n

$$\Psi_n(z) < 0 \quad \text{if} \quad z \in N_n$$

and therefore

$$\psi(z) \leq 0 \quad \text{if} \quad z \in \overline{N}.$$

Hence again by the maximum - principle

(12) $$\psi(z) < 0 \quad \text{if} \quad z \in N.$$

2.6. Estimates on the vertical spreading of the domain $Q[u]$. (Open flows)

If $P = \Delta - Q$ is simply connected then it is known that the image $\hat{Q} = g(Q)$ has finite height if $\tau'[Q] = \tau[P]$ is bounded (cf. Garabedian-Spencer (1952) p. 382). This is, however, not necessarily true if P is not simply connected. Unfortunately we have to admit as admissible domains Q any closed subsets of Δ. The measure of the subset $Q_h = \left\{\zeta : \zeta \in Q, \ \eta > h\right\}$ can be expected nevertheless to tend to zero as $h \to \infty$ for any closed $Q \subset \Delta$, if $\tau'[Q]$ is finite. We will estimate, therefore, the minimum of $\tau'[Q]$ if

$$\text{mes } Q_h = A_h > 0.$$

$\tau'[Q]$ will not increase if $Q[\psi]$ is replaced by Q_h, because ψ is an admissible function for the minimum-problem of the functional T in the domain $E - Q_h$. It can be also assumed that Q_h is symmetric to the imaginary axis, since Steiner-symmetrization will not increase $T[\psi]^*$, and leaves mes Q_h unchanged. The value of τ' is further reduced (or not increased) by replacing Q_h by its intersection R with the imaginary axis, because of the monotonic dependence of $\mathcal{T}$ on the domain, hence the inequality

(1) $$\tau'[Q] \geq \tau'[R].$$

---

* cf. Pólya-Szegö (1951).

(2.6)

On the other hand, suppose that (after symmetrization) Q_h contains a horizontal segment $I = (ik - a/2, ik + a/2)$, $(k > h)$. Then similarly

(2) $$\tau'[Q] \geq \tau'[I] .$$

We will estimate now $\tau'[R]$ and $\tau'[I]$.

(a) If $\psi_1(z)$ is the solution of the outer problem in E-I, $\psi^* = \hat{\psi}_1 - \eta$, then by Schwarz's inequality

$$\int_0^k \left(\frac{\partial\psi^*}{\partial\eta}\right)^2 d\eta \geq \frac{1}{k}(\psi^*)_I^2 = k \ .$$

Hence

(3) $$\tau'(I) \geq \int_{-a/2}^{a/2} d\xi \int_0^{\pi} \left(\frac{\partial\psi^*}{\partial\eta}\right)^2 d\eta \geq ka \geq ha \ .$$

(b) Similarly, if $\hat{\psi}_2(\zeta)$ is the solution of the outer problem of R, $\psi^*(\zeta) = \hat{\psi}_2(\zeta) - \eta$ and $\zeta = r\,e^{i\theta}$, $\psi^*(\zeta) = \Psi^*(r,\theta)$, then

(4) $$\int_0^{\pi/2} \left(\frac{\partial\Psi}{\partial\theta}\right)^2 d\theta \leq \frac{2}{\pi}\left(\int_0^{\pi/2} \frac{\partial\Psi}{\partial\theta}\, d\theta\right)^2 = \frac{2}{\pi}\,\Psi\,(ir)^2 = \frac{2}{\pi}\,r^2.$$

Hence, let S denote the set obtained by rotating R around the origin. Then using the estimate (4),

$$\tau'[Q_h'] \geq \iint_S \left(\frac{\partial\Psi}{\partial\theta}\right)^2 \frac{dr}{r}\, d\theta \geq \frac{4}{\pi}\int_R r\, dr \geq \frac{4}{\pi}\int_h^{h+b} r\, dr$$

or

(5) $$\tau'[Q_h'] > \frac{4}{\pi}\, h\, b$$

where b is the linear measure of R. From (1), (2), (3), and (5)

$$\tau'[Q]^2 \geq \frac{4}{\pi} h^2 ab .$$

If "mes_1" denotes linear measure, then

$$ab = a\,\text{mes}_1\, R \geq \text{mes}\, Q_h = A_h.$$

Thus we find for open flows

$$\text{(6)} \qquad T[u]^2 \geq \tau'[Q]^2 \geq (4/\pi)\, h^2\, A_h .$$

2.7. An all-important point in the proof of the existence ofa solution of the minimum problems will be to show that if $\{\psi_n\}$ is a sequence of admissible functions it is not possible that $\psi_n(z) \to 0$ in all fixed finite domains. This will be achieved by lemma 2.8, which will essentially state that if Dirichlet-integral and the area of support of a function vanishing on the real axis are finite, then the function values in a strip adjoining the real axis are in a sense lumpy. For the proof we first need an inequality of the type (1.4.1), but with milder assumptions.

Lemma 2.7. Assume that the function $u(z)$ defined over the rectangle

$$R = (0,s) \times (0,h)$$

vanishes on the lines $y = 0$, $y = h$, is continuous and has a finite support area A and its Dirichlet integral $D = D[u]$ is finite. Then D, A and $L = L[u]$ satisfy the inequality

$$\text{(1)} \qquad L \leq (8/\pi)^{\frac{1}{2}}\, (1 + h/\pi s)\, A D^{\frac{1}{2}} .$$

Proof. We write

$$u(z) = v(z) + w(z), \quad v(z) = (x/s)\, u(z).$$

By Schwarz's inequality

$$(2) \qquad D[v] = (1/s^2)\left\{\iint x^2 (\nabla u)^2 \, dx\, dy + 2\iint x\, u \frac{\partial u}{\partial x}\, dx\, dy + \iint u^2 \, dx\, dy\right\}$$

$$\leq \left\{D[u]^{\frac{1}{2}} + (1/s)\left(\iint u^2 \, dx\, dy\right)^{\frac{1}{2}}\right\}^2 .$$

On the other hand

$$k\int_0^h u^2 \, dy \leq \int_0^h \left(\frac{\partial u}{\partial y}\right)^2 dy$$

where $k = (\pi/h)^2$ is the smallest eigenvalue of the vibrating string of length h. Substituting this into (2) yields

$$(3) \qquad D[v] \leq (1 + h/\pi s)^2 \, D[u] .$$

Let us continue v(z) into the rectangle

$$R^* = (s, 2s) \times (0, h)$$

by reflection on the line $x = s$. Then v is vanishing on the boundary of the domain d consisting of the support of u and its mirror - image on the line $x = s$. d has area 2a. Hence by the inequality (1.4.1) and by (3) and (4)

$$|\, L[v] \,| \leq (2/\pi)^{\frac{1}{2}} (1 + h/\pi s)\, A D^{\frac{1}{2}}$$

and similarly

$$|\, L[w] \,| \leq (2/\pi)^{\frac{1}{2}} (1 + h/\pi s)\, A D^{\frac{1}{2}} .$$

Adding these inequalities yields

$$|\, L[u] \,| \leq 2(2/\pi)^{\frac{1}{2}} (1 + h/\pi s)\, A D^{\frac{1}{2}} ,$$

which is equivalent to (1).

2.8. Lemma. Let I denote any interval of length ℓ, $H = (0, h)$.

(2.8)

Suppose the function u defined in the rectangle domain $R = I \times H$ has there a finite Dirichlet - integral D, vanishes on the lines $y = 0$, $y = h$, and its support has area A. Then given any s, $0 < s < \ell$, there is a subinterval $I^* \subset I$ of length exceeding s, such that

$$(1) \qquad L^* \geq \frac{\pi}{8} \frac{1}{(1 + 2h/\pi s)^2} \frac{L^3}{A^2 D}$$

where

$$L = L[u \mid I \times H], \quad L^* = L[u \mid I^* \times H] .$$

<u>Proof*</u> Suppose that

$$(n - 1)\, s < \ell \leq ns$$

where n is an integer > 1. Then I can be subdivided into n parts (I_1, I_2, ..., I_n) of equal length σ where

$$(2) \qquad \begin{aligned} &s/2 < \sigma \leq s \\ &L_k = L[u \mid I_k \times H] , \\ &L^* = \max \left\{ L_k : k = 1, \ldots, n \right\} \\ &D_k = D[u \mid I_k \times H] \end{aligned}$$

and A_k the area of the support of u in the rectangle $I_k \times H$. By Lemma 2.7 then

$$L_k \leq (8/\pi)^{\frac{1}{2}} (1 + h/\pi\sigma)\, A_k D_k^{\frac{1}{2}} \equiv \alpha\, A_k D_k^{\frac{1}{2}} .$$

By Hölder's inequality then

$$\sum_{k=1}^{n} L_k^{2/3} \leq \alpha^{2/3} \sum_{k=1}^{n} A_k^{2/3} D_k^{1/3} \leq \alpha^{2/3} \Big(\sum_{k=1}^{n} A_k\Big)^{2/3} \Big(\sum_{k=1}^{n} D_k\Big)^{1/3}$$

or

---

* The method of this proof is due to Professor Peter D. Lax (unpublished communication). The author proved only a weaker result (which however is still satisfactory for the present purpose) and not as elegantly.

$$(3) \qquad L = \sum_{k=1}^{n} L_k \le (L^*)^{1/3} \sum_{k=1}^{n} L_k^{2/3} \le (L^*)^{1/3} \alpha^{2/3} A^{2/3} D^{1/3}.$$

Noting that by (2)

$$\alpha = (8/\pi)^{\frac{1}{2}} (1 + h/\pi\sigma) \ge (8/\pi)^{\frac{1}{2}} (1 + 2h/\pi s)$$

and introducing this inequality into (3) we get (1).

Remarks. (i) Since the length ℓ does not occur explicitly in (1), the latter remains valid even if I is semi-infinite or infinite.

(ii) The inequality (1) remains valid even if $u(x + iy)$ is discontinuous on the lines $x = x_1, x_2, \ldots, x_k$, $(x_o < x_1 < \ldots < x_k < x_{k+1}$; $I = (x_o\, x_{k+1})$), if

$$s < \min \{ x_\nu - x_{\nu-1};\ \nu = 1, \ldots, k+1 \} .$$

(iii) The requirement $u(x + ih) = 0$ (x real) is not essential. If only $u(x) = 0$ is assumed, than by reflection on the line $y = h$ the inequality

$$(4) \qquad L^* \ge \frac{\pi}{16} \frac{1}{(1 + 4h/\pi s)^2} \frac{L^3}{A^2 D}$$

can be derived from the reinterpretation of (1).

2.9 Convergence of the minimizing sequence.

Suppose that the minimum problem III' or IV is wellposed for some given Λ, λ, and m or ω respectively. Then there is a sequence of functions ψ_ν of $\mathcal{J}$, such that:

(2.9)

$$V'[\psi_\nu] \to \inf \{V'[u] : u \in \mathcal{J}\} = J_{III}$$

and

$$W[\psi_\nu] \to \inf \{W[u] : u \in \mathcal{J}\} = J_{IV}$$

respectively.

It will be shown that the sequence $\{\psi_\nu\}$ contains a subsequence which converges pointwise to an admissible function ψ, which is the solution of the minimum-problem. We also show that $\psi \in \mathcal{H}\mathcal{S}$.

(i) We start with the proof of the existence of a subsequence converging to a function $\psi(z) \in \mathcal{J}\,\mathcal{H}\mathcal{S}$. Steiner symmetrization with respect to the imaginary axis of the time ζ-plane does not increase the value of the functionals V' or W. In fact, for any $u \in \mathcal{D}$, Steiner-symmetrization yields a function $\hat{u}_s(\zeta) \in \mathcal{S}$, and $\hat{N}_s = N[\hat{u}_s]$. The Dirichlet-integral is not increased by Steiner-symmetrization.* Hence with obvious notations

(1) $$D[(\hat{u}_s)_+ - \eta] \leq D[\hat{u}_+ - \eta],$$

(2) $$D[(\hat{u}_s)_-] \leq D[\hat{u}_-].$$

Furthermore from

$$L[u_-] = -2 \iint_{\hat{N}} |f'(\zeta)|^2 \hat{u}(\zeta)\, d\xi\, d\eta ,$$

$$A[u] = \iint_{\hat{N}} |f'(\zeta)|^2\, d\xi\, d\eta$$

* See Pólya-Szegö (1951)

(2.9)

it is clear that

(3) $$L\left[(u_s)_-\right] \geq L\left[u_-\right], \quad A[u_s] \geq A[u]$$

since $|f'(\zeta)|$ is a non-increasing function of ξ. Hence combining Eqs. (1), (2), (3) according to Eq. (1.2.7),

(4) $$V'[u_s] \leq V'[u],$$

(5) $$W[u_s] \leq W[u].$$

Now suppose that the solution of the restricted minimum-problem defined by the functions sets $P^* = P[u_s]$, $N^* = N[u_s]$ is the function $v(z) \in \mathcal{J}\mathcal{H}$. Then we claim that $v \in \mathcal{S}$ as well. In fact, if this were not true, then symmetrization to the imaginary ζ - axis would reduce the values of the functionals V' and W, leaving the already symmetrized domains P^*, N^* unchanged. But this would contradict the assumption that $v(\zeta)$ is a solution of the restricted minimum-problem defined by P^* and N^*. Therefore by (4), (5)

$$V'[v] \leq V'[u],$$
$$W[v] \leq W[u]$$

where $v \in \mathcal{J}\mathcal{H}\mathcal{S}$. It is no restriction of generality therefore to assume that

$$\psi_n \in \mathcal{J}\mathcal{H}\mathcal{S}$$

to begin with.

The supremum of the values $T[\psi_n]$ depends by the remark in Section 1.6 only on the supremum of the values $V'[\psi_n]$, $W[\psi_n]$ respectively. Therefore there is a positive j, such that

$$T[\psi_n] < j \quad (n = 1, 2, \ldots)$$

(2.9)

Hence the sequence $\{\psi_n\}$ meets the requirements of 2.2, consequently the functions $\hat{\psi}_\nu(\zeta)$ are equicontinuous. By Arzelà's theorem therefore, they contain a subsequence converging to a continuous $\hat{\psi}(\zeta)$. Since each $\psi_\nu(z)$ belongs to $\mathcal{S}$, so does $\psi(z)$. We maintain that $\psi(z) \in \mathcal{H}$. Let $\zeta_1 \in P[\hat{\psi}]$. Since $P[\hat{\psi}]$ is open, ζ_1 has a neighborhood $\sigma_\varepsilon = \{\zeta : |\zeta - \zeta_1| < \epsilon\}$ in $P[\hat{\psi}]$. Suppose that

$$\hat{\psi}(\zeta) \geq \delta(\epsilon) \quad \text{in } \sigma_\varepsilon .$$

Then, because of the equicontinuity of the functions $\hat{\psi}_\nu$, there is an integer ν_o such that

$$\hat{\psi}_\nu(\zeta) > \tfrac{1}{2}\,\delta(\epsilon) \quad \text{in } \sigma_\epsilon ,$$

and therefore

$$\sigma_\varepsilon \in P[\psi_\nu] \quad \text{for} \quad \nu \geq \nu_o .$$

Then by the theory of normal families, $\hat{\psi}(\zeta)$ is harmonic in σ_ε. Since ζ_1 was arbitrary, $\hat{\psi}(\zeta)$ is harmonic everywhere in $P[\hat{\psi}]$. If $\zeta_1 \in N[\hat{\psi}]$ together with a neighborhood σ_ϵ as above, then similarly $\sigma_\epsilon \subset N[\hat{\psi}_\nu]$ for $\nu > \nu_1$, and $\psi_\nu(z)$ satisfies

$$\nabla^2 \psi_\nu(z) = \omega_\nu \quad \text{in} \quad f(\sigma_\varepsilon) = \tau_\epsilon .$$

In case of IV, $\omega_\nu = \omega$. In case of III'

$$\omega_\nu = (2/m)\, D[\psi_\nu] \leq (2/m) \max_\nu T[\psi_\nu] \leq 2j/m .$$

We can select therefore a subsequence such that

$$\omega_{\nu_k} \to \omega .$$

Again only the subsequence is kept, and ψ_{ν_k} relabeled ψ_k. The functions

$$\psi_\nu^*(z) = \psi_\nu(z) - (\omega_\nu/2)\, y^2$$

(2.9)

are harmonic and uniformly bounded in σ_ϵ, and therefore again

$$\psi_\nu^*(z) \to \psi^*(z) = \psi(z) - (\omega/2)\, y^2$$

uniformly in σ_ϵ, and $\psi^*(z)$ is harmonic there. Therefore the function $\psi(z)$ will satisfy

$$\nabla^2\, \psi(z) = \omega$$

in σ_ϵ, consequently everywhere in $N[\psi]$. By the lower semicontinuity of the Dirichlet-integral

$$D\left[\hat{\psi}_+ - \eta\right] \le \underline{\lim}_{n\to\infty}\; D\left[\left(\hat{\psi}_n\right)_+ - \eta\right]$$

$$D\left[\hat{\psi}_-\right] \quad \le \underline{\lim}_{n\to\infty}\; D\left[\left(\hat{\psi}_n\right)_-\right]$$

Combining these inequalities and considering Eq. (1.2.7), we find

(7) $$T[\psi] \le \underline{\lim}_{n\to\infty}\; T[\psi_n]$$

which implies of course that $\psi \in \mathcal{J}$.

(ii) We show that

(9) $$\lim_{n\to\infty} L\left[\left(\psi_n\right)_-\right] = L\left[\psi_-\right]$$

If Ω is any bounded open domain then certainly

(10) $$L\left[\left(\psi_n\right)_- \,\middle|\, \Omega\right] \to L\left[\psi_- \mid \Omega\right]$$

because of the uniform convergence $\psi_\nu \to \psi$ in Ω. Let us choose a fixed $h > 0$ (for channel flows we choose $h = \pi$), and for arbitrary p, q ($p < q$) introduce $\tilde{R}_p^q = f(R_p^q)$ where R_p^q is the rectangular domain

$$R_p^q = (p,q) \times (0,h)\ .$$

Then (10) is valid for $\Omega = \tilde{R}^q_p$. We show next that it remains valid even if $\Omega = R^{+\infty}_{-\infty} = S_h$, (which is unbounded). First it will be shown that for the elements ψ_n of the minimum-sequence

(11) $$\lim_{a\to\infty} \lim_{n\to\infty} L\left[(\hat{\psi}_n)_- \mid R^{\infty}_a\right] = 0.$$

By lemma 2.8 (Eq.(2.8.4)) there is an interval $(c, c + 4h/\pi) \subset (a, \infty)$ such that

(12) $$L_n^* \geq K L_n^3$$

where

(12') $$L_n = L\left[(\psi_n)_- \mid R^{\infty}_a\right],\ L_n^* = L\left[(\psi_n)_- \mid R_c^{c+4h/\pi}\right],$$

and

$$K = \pi/64\ j^3 .$$

In the determination of the constant K it was taken into consideration that

$$j = \sup_n T[\psi_n]$$

is an upper bound of both $D\left[(\psi_n)_-\right]$ and $A[\psi_n]$.

Since the function $\hat{\psi}_n$ is symmetrized to the η - axis, the lefthand side of (12) can only increase if c is replaced by a in (12').

Suppose that (11) is not true. Then there is a number b, such that for any a > there is a subsequence $\{\psi_{n_k}\}$ such that

$$L\left[\left(\hat{\psi}_{n_k}\right)_- \mid R^{\infty}_a\right] > t > 0 \quad (k = 1, 2, \ldots)$$

Then by (12)

$$L\left[\left(\hat{\psi}_{n_k}\right)_- \mid R_a^{a+4h/\pi}\right] > K t^3 .$$

The functions $\hat{\psi}_n(\zeta)$ have the uniform limit $\hat{\psi}(\zeta)$ in $R_a^{a+4h/\pi}$. Therefore we must also have for any $a > b$, where b is fixed,

$$L\left[\hat{\psi}_- \mid R_a^{a+4h/\pi}\right] \geq K t^3 > 0.$$

However, this is impossible for bounded $L[\hat{\psi}_-]$. Thus (11) is proved.

Let ϵ be any given positive number. We choose then an a_1 such that for $a > a_1$

$$(13) \qquad L\left[\hat{\psi}_- \mid R_a^{\infty}\right] < \epsilon .$$

Then we select an $a > a_1$ and a ν such that

$$(14) \qquad L\left[(\hat{\psi}_n)_- \mid R_a^{\infty}\right] < \epsilon$$

if $n > \nu$. (This selection is possible by (11)). Then, fixing the value of a, we select an $n \geq \nu$ such that

$$(15) \qquad \left| L\left[\left\{(\hat{\psi}_n)_- - \hat{\psi}_-\right\} \mid f'(\zeta) \mid^2 \middle| R_{-a}^{a}\right]\right| < \mu^2 \epsilon$$

where $\mu = \sup\left\{|f'(\zeta)| : \zeta \in E\right\}$. This is possible by the uniform convergence of $\hat{\psi}_n$ to $\hat{\psi}$ in R_{-a}^{a}. From (13), (14)

$$L\left[\left\{(\hat{\psi}_n)_- - \hat{\psi}_-\right\} \mid f'(\zeta) \mid^2 \middle| R_a^{\infty}\right] < \mu^2 \epsilon .$$

In combination with (15) therefore

$$(16) \qquad \begin{cases} L\left[(\psi_n)_- - \psi_- \mid f(S_h)\right] \\ \qquad = L\left[\mid f'(\zeta) \mid^2 \left\{(\hat{\psi}_n)_- - \hat{\psi}_-\right\} \middle| S_h\right] \leq 3\mu^2 \epsilon \end{cases}$$

where $S_h = R_{-\infty}^{\infty}$. We note that (16) is valid for any fixed h, if only n is sufficiently large. For channel flows with the choice $h = \pi$, (16) completes the proof of

(17) $$L\left[(\psi_m)_-\right] \to L\left[\psi_-\right].$$

In case of open flows, we select a sequence $h_n \to \infty$. Then by Eq. (2.6.6) the measure A_{h_m} of the sets $\Sigma_{mn} = \left\{\zeta : \hat{\psi}_n(\zeta) < 0 ; \eta > h_m\right\}$ converges to zero as $m \to \infty$, uniformly in n. Since the functions $\psi_n(z)$ are uniformly bounded in $N[\psi_n]$, also

$$L\left[(\hat{\psi}_n)_- \mid \Sigma_{mn}\right] \to 0$$

uniformly in n, hence we can select m so large that

$$\left|L\left[(\hat{\psi}_n)_- - \hat{\psi}_- \mid \Sigma_{mn}\right]\right| < \epsilon ,$$

or

(18) $$\left|L\left[(\psi_n)_- - \psi_- \mid f(\Sigma_{mn})\right]\right| < \mu^2 \epsilon .$$

Thus, selecting $h = h_m$ in (16), and combining with (18),

$$\left|L\left[(\psi_n)_- - \psi_-\right]\right| \leq 4\mu^2\epsilon ,$$

which is equivalent to (17).

(iii) We will show that for any bounded measurable set S

(19) $$\overline{\lim_{n \to \infty}} \text{ mes } \left\{ S \cap Q[\psi_n]\right\} \leq \text{mes } \left\{S \cap Q[\psi]\right\} .$$

(2.9)

Notations:

$$S \cap Q[\psi_\nu] = R_\nu, \quad S \cap Q[\psi] = R, \quad M_n = \bigcup_{\nu=n}^{\infty} R_\nu, \quad \rho = \bigcap_{n=1}^{\infty} M_n.$$

The set ρ contains all points $z^* \subset S$ such that $\psi_n(z^*) \leq 0$ for infinitely many values of n. For such z^* then $\psi(z^*) \leq 0$, hence

(20) $$\rho \subset R .$$

On the other hand since the nested sets M_n are bounded

$$\text{mes } \rho = \lim_{n \to \infty} \text{mes } M_n$$

hence from $R_n \subset M_n$

$$\text{mes } \rho \geq \overline{\lim_{n \to \infty}} \text{ mes } R_n .$$

Combining this with (20), we get

$$\overline{\lim_{n \to \infty}} \text{ mes } R_n \leq \text{mes } R ,$$

which is identical with (19).

(19) remains valid if $g(S)$ is the strip $\hat{S}_a^b = \{\zeta : a < \xi < b\}$.

For channel flows this is no new statement. For open flows we set

$$\Sigma_h = (a, b) \times (0, h).$$

Then by Eq. (2.6.6) we can select h so large that

$$\text{mes } \{ Q[\hat{\psi}_n] \cap (\Sigma_\infty - \Sigma_h) \} < \epsilon, \quad (\Sigma_\infty = S_a^b)$$

h independent of n. Thus by application of (19) to Σ_h

$$\overline{\lim_{n\to\infty}} \text{ mes}\left(\Sigma_\infty \cap Q[\hat{\psi}_n]\right) \leq \overline{\lim_{n\to\infty}} \text{ mes}\left(\Sigma_h \cap Q[\hat{\psi}_n]\right) + \epsilon$$
$$\leq \text{mes}\left(\Sigma_h \cap Q[\hat{\psi}]\right) + \epsilon \leq \text{mes}\left(\Sigma_\infty \cap Q[\hat{\psi}]\right) + \epsilon \tag{2.9}$$

is obtained. Since ϵ is an arbitrary positive number, (19) is valid for $S_a^b = \Sigma_\infty$ as well.

(iv) After these preliminaries we are now in the position to prove that

$$V'[\psi] \leq \underline{\lim_{n\to\infty}} V'[\psi_n] \tag{III'}$$

or

$$W[\psi] \leq \underline{\lim_{n\to\infty}} W[\psi_n] . \tag{IV}$$

We use the notations

$$N_a^b[u] = \{z : z \in N[u] ; a < \text{Re } g(z) < b\},$$

$P_a^b[u]$, $Q_a^b[u]$ defined analogously ;

$$L_a^b[u] = -2 \iint_{N_a^b[u]} u(z)\, dx\, dy = -2 \iint_{a<\xi<b} \hat{u}_-(\zeta) \,|\, f'(\zeta)\,|^2 \,d\xi\, d\eta ,$$

$$A_a^b[u] = \text{mes}\left(\Delta - P_a^b[u]\right) = \text{mes } Q_a^b[u]$$
$$= \iint_{\hat{u}\leq 0;\, a<\xi<b} |\, f'(\zeta)\,|^2 \,d\xi\, d\eta ;$$

$$T_a^b[u] = \iint_{a<\xi<b} \left(\nabla[\hat{u}(\zeta) - \eta]\right)^2 d\xi\, d\eta .$$

(2.9)

Then with the notation $\mu_a = \sup \{ |f'(\zeta)| : a < \xi \}$,

(21) $$A_a^\infty [\psi_n] \leq \mu_a^2 \iint_{\xi > a,\ \hat{\psi}_n < 0} d\xi\, d\eta = \mu_a^2 A_a^\infty [\hat{\psi}_n] .$$

By the well-known semicontinuity property of the Dirichlet-integral

(22) $$T_{-a}^a [\psi_+] \leq \underline{\lim}_{n \to \infty} T_{-a}^a \left[(\psi_n)_+ \right] .$$

On the other hand,

(23) $$T_a^\infty \left[(\psi_n)_+ \right] \geq A_a^\infty \left[\hat{\psi}_n \right] .$$

This is a direct consequence of (1.2.8), if applied to the function

$$\hat{\psi}_n^* (\zeta) = \begin{cases} \hat{\psi}_n(\zeta) & \text{if } \xi \geq a \\ \hat{\psi}_n(2a - \bar{\zeta}) & \text{if } \xi < a . \end{cases}$$

From (23) then by (21)

(24) $$T_a^\infty \left[(\psi_n)_+ \right] \geq \left(1/\mu_a^2 \right) A_a^\infty [\psi_n] .$$

Combining (23), (24),

$$T_{-a}^a [\psi_+] \leq \underline{\lim}_{n \to \infty} \left\{ T\left[(\psi_n)_+ \right] - \left(2/\mu_a^2 \right) A_a^\infty [\psi_n] \right\} ,$$

or by (21)

$$T_{-a}^a [\psi_+] - \lambda A_{-a}^a [\psi] \leq \underline{\lim}_{n \to \infty} \left\{ T\left[(\psi_n)_+ \right] - \lambda A [\psi_n] - 2(\mu_a^{-2} - \lambda) A_a^\infty [\psi_n] \right\} .$$

For $a \to \infty$, $\mu_a \to \alpha^{-2} = \Lambda$, and since $\lambda \leq \Lambda$, we get

$$(25) \qquad T[\psi_+] - \lambda A[\psi] \leq \varliminf_{n\to\infty} \left\{ T[(\psi_n)_+] - \lambda A[\psi_n] \right\} .$$

Also by the lower semicontinuity of the Dirichlet-integral,

$$(26) \qquad D[\psi_-] \leq \varliminf_{n\to\infty} D[(\psi_n)_-]$$

Combining (17), (25), (26) we find (with obvious indices III, IV)

$$V'[\psi_{III}] \leq \varliminf_{n\to\infty} V'[\psi_n] = J_{III} ;$$

$$W[\psi_{IV}] \leq \varliminf_{n\to\infty} W[\psi_n] = J_{IV} .$$

Since ψ itself is admissible for III$'$ and IV respectively,

$$V'[\psi_{III}] = J_{III} , \quad W[\psi_{IV}] = J_{IV} .$$

2.10 Continuous dependence of solutions on the domain and on λ, ω, m.

Theorem. Suppose that the sequence of domains $\{\Delta_n\}$ is characterized by a sequence of functions $\{f_n(\zeta)\}$ convergent in E in the Dirichlet-norm to a function $f(\zeta)$ defining the admissible domain Δ. Let $Z_{III}(\delta)$, $Z_{IV}(\delta)$ denote the set of all solutions of the problems III$'$, IV respectively in the domain δ. Let Ω be any open bounded set which with its closure is a subset of Δ, and $\mathcal{L}_\infty[\Omega]$ the metric space with the distance

$$\rho(u, v) = \sup_{z \in \Omega} | u(z) - v(z) | .$$

Then for any Ω,

$$Z(\Delta_n) \to Z(\Delta)$$

(2.10)

in the $\mathcal{L}_\infty[\Omega]$ metric. $(Z = Z_{III}, Z_{IV})$.

Proof. We have to show that if $\{\psi_n(z)\}$ is a sequence of solutions in Δ_n, then it contains a subsequence $\{\psi_{k_n}(z)\}$ converging to a solution in Δ, uniformly in any Ω. There certainly is a convergent subsequence as shown in the preceding section. Therefore it only has to be shown that the limit $\psi(z)$ of this subsequence (uniform limit in any Ω) is a solution of the corresponding minimum-problem for Δ. It is no restriction of generality to assume that the original sequence itself converges to ψ. Then by the semi-continuity of the Dirichlet integral

(1) $$T[\psi] \leq \underline{\lim}_{n\to\infty} T[\psi_n]$$

whereas

(2) $$L[\psi_-] = \lim_{n\to\infty} L\left[(\psi_n)_-\right],$$

(3) $$A[\psi] \geq \overline{\lim}_{n\to\infty} A[\psi_n] .$$

By (1), (2), (3) therefore

(4) $$V'[\psi] \leq \underline{\lim}_{n\to\infty} V'[\psi_n] ,$$

(5) $$W[\psi] \leq \underline{\lim}_{n\to\infty} W[\psi_n] .$$

Suppose that $\psi(z)$ is not a solution of the minimum-problem corresponding to Δ. Then any solution $\psi^*(z)$ satisfies the inequality

(6/III) $$B \equiv V'[\psi] - V'[\psi^*] > 0 \qquad \text{(III')}$$

or

(6/IV) $$C \equiv W[\psi] - W[\psi^*] > 0 \qquad \text{(IV)}.$$

(2.10)

The functions

$$h_n(z) = f_n\left(g(z)\right) \qquad (g = f^{-1})$$

which map the domain Δ into Δ_n may be used to construct admissible functions of the minimum-problem corresponding to the domain Δ_n. Thus

$$\psi_n^*(z) = \psi^*\left(h_n(z)\right)$$

are competing functions in Δ_n. By definition we have

(7) $$T\left[\psi_n^*\right] \equiv D\left[\hat{\psi}_n^* - \eta\right] \equiv T[\psi],$$

further

(8) $$L\left[(\psi_n^*)_-\right] = -2\iint\limits_{\hat{\psi}<0} |f_n'(\zeta)|^2\,\hat{\psi}^*(\zeta)\,d\xi\,d\eta$$

$$= -2\iint\limits_{N[\psi]} |h_n'(z)|^2\,\psi^*(z)\,dx\,dy$$

and similarly

(9) $$A\left[\psi_n^*\right] = -2\iint\limits_{N[\psi]} |h_n'(z)|^2\,dx\,dy.$$

From the hypothesis that $f_n(\zeta) \to f(\zeta)$ in the Dirichlet - norm, follows that $h_n(z) \to (z)$ in the same sense. Therefore by (8) and (9)

(10) $$\lim_{n\to\infty} L\left[(\psi_n^*)_-\right] = L\left[\psi_-^*\right],$$

(11) $$\lim_{n\to\infty} A\left[\psi_n^*\right] = A\left[\psi^*\right].$$

(2.10)

By (7), (10), (11) a number ν exists to any given $\epsilon > 0$ such that for any $n \geq \nu$

(12) $$V'\left[\psi_n^*\right] \leq V'\left[\psi^*\right] + \epsilon, \quad W\left[\psi_n^*\right] \leq W\left[\psi^*\right] + \epsilon .$$

On the other hand, by (4) and (5) a number $m \geq \nu$ exists such that

(13) $$V'[\psi] < V'\left[\psi_m\right] + \epsilon, \quad W[\psi] < W\left[\psi_m\right] + \epsilon .$$

Combining (12) and (13) and (6) we obtain:

(14) $$V'\left[\psi_m^*\right] < V'\left[\psi_m\right] - B + 2\epsilon , \qquad (B > 0),$$

(15) $$W\left[\psi_m^*\right] < W\left[\psi_m\right] - C + 2\epsilon , \qquad (C > 0).$$

(14) or (15) are valid for any given ϵ for some m. If we set therefore $2\epsilon < \min(B, C)$, we get a contradiction to the assumption that ψ_m is the solution of the minimum-problem III$'$ or IV corresponding to Δ_m. Hence ψ must be a solution of the minimum-problem III$'$ or IV for Δ.

<u>The solution-set Z depends continuously on the parameters λ, and ω or m.</u> Suppose that $\psi_n(z)$ is a solution of III$'$ or IV for $\lambda = \lambda_n \to \lambda_\infty$, ω or m and Δ fixed (λ_∞ must be such that the problem remains well-posed with $\lambda = \lambda_\infty$). Then by the argument already used in this section we may assume that $\psi_n(z) \to \psi(z)$ where $\psi(z)$ is admissible. Therefore also

(16) $$V'\left[\psi_n ; \lambda_n\right] \to V'\left[\psi ; \lambda_\infty\right], \quad W\left[\psi_n ; \lambda_n\right] \to W\left[\psi ; \lambda_\infty\right]$$

where the dependence of the functionals V', W on λ is made explicit. On the other hand for any admissible u,

(2.10)

$$(17/\mathrm{III}) \qquad V'[u ; \lambda_n] \geq V'[\psi_n ; \lambda_n]$$

and

$$(17/\mathrm{IV}) \qquad W[u ; \lambda_n] \geq W[\psi_n ; \lambda_n]$$

respectively. Combining the relations (16), (17) and the assumption $\lambda_n \to \lambda_\infty$, we find

$$V'[u ; \lambda_\infty] \geq V'[\psi ; \lambda_\infty], \quad \text{and} \quad W[u ; \lambda_\infty] \geq W[\psi ; \lambda_\infty]$$

respectively. Hence ψ is a solution of the minimum problem for $\lambda = \lambda_\infty$.

The continuous dependence on ω or m can be shown similarly.

Part III

TOPOLOGICAL PROPERTIES OF THE SOLUTION

3.1 Theorem. The sets $\beta \cup \overline{N[\psi]}$ and $\Gamma = \partial P[\psi] - \beta'$ are connected*.

Proof. To study the connectedness of the set N, we will show that if $\beta \cup \overline{N}$ consists of two closed unconnected sets, O_1 and O_2 and if we replace these with solid bodies free to move, the pressure forces resulting from the potential flow will force them together, accompanied by a decrease of virtual mass.

More precisely, let O_1 and O_2 denote two disjoint closed sets, such that the admissible outer streamfunction $\psi_+(z)$ vanishes in $O_1 \cup O_2 \equiv O[\psi^+]$ and is harmonic in $\Delta_1 = \Delta - O[\psi_+]$, and that O_1 is bounded away from β. We assume that O_1 is free to move without rotation in Δ. If c is a complex number denoting a point fixed to O_1, then $T[\psi_+]$ is a function of c, say, $T[\psi_+] = \tau(c, \overline{c})$.** We will compute the first and second derivatives of $\tau(c, \overline{c})$.

We will first assume that O_1 is bounded by an analytic curve γ_1. Let us denote $\psi_+(z, \overline{z}) \equiv \psi_+(z, \overline{z}; c, \overline{c})$. We will need the derivative $\partial\psi_+/\partial\overline{c}$. Since γ_1 was assumed analytic, we may continue ψ_+ analytically (as a harmonic function) beyond γ_1 into a strip of width everywhere exceeding $\delta > 0$. Let us choose c^* such that $|c^* - c| < \delta$, and denote $\psi_+(z, \overline{z}, c^*, \overline{c}^*) = \psi_+^*(z, \overline{z})$. If $|c^* - c|$

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* In case of open flows β' is empty.

** For sake of clarity in this section h(z) will denote an analytic function, $k(z, \overline{z})$ any function of the complex variable z.

(3.1)

is small, then

(1) $$\psi^*(z,\bar z)\Big|_{z\in\gamma_1} = -\left[\frac{\partial\psi_+}{\partial z}\right]_{z\in\gamma_1}(c^*-c) + O(|c^*-c|^2).$$

The function $\psi_+^*(z,\bar z) - \psi_+(z,\bar z)$ can be continued as a harmonic function in Δ_1, and $D[\psi^*-\psi|\Delta_1]<\infty$. Hence denoting Green's function for the domain Δ_1 by $G(z,\bar z,\zeta,\bar\zeta)$,

$$\psi^*(z,\bar z) - \psi(z,\bar z) = -(c^*-c)\int_{\gamma_1}\frac{\partial G(z,\bar z;\zeta,\bar\zeta)}{\partial n_\zeta}\,\frac{\partial\psi(\zeta,\bar\zeta)}{\partial\zeta}|d\zeta| + O(|c^*-c|^2)$$

$$= 2i(c^*-c)\int_{\gamma_1}\frac{\partial G(z,\bar z,\zeta,\bar\zeta)}{\partial\zeta}\,\frac{\partial\psi(\zeta,\bar\zeta)}{\partial\zeta}\,d\zeta + O(|c^*-c|^2).$$

Letting $c^*\to c$, we find

$$\frac{\partial\psi}{\partial c} = 2i\int_{\gamma_1}\frac{\partial G(z,\bar z;\zeta,\bar\zeta)}{\partial\zeta}\,\frac{\partial\psi}{\partial\zeta}\,d\zeta,$$

and similarly

(2) $$\frac{\partial\psi}{\partial\bar c} = -2i\int_{\gamma_1}\frac{\partial G(z,\bar z;\zeta,\bar\zeta)}{\partial\bar\zeta}\,\frac{\partial\psi}{\partial\bar\zeta}\,d\bar\zeta.$$

We find from (1.3.4) and (1) that

$$T[\psi^*] - T[\psi] = -2i\int_{\gamma_1}(\psi^*-\psi)\frac{\partial\psi}{\partial z}\,dz + O(|c^*-c|^2)$$

$$= 2i(c^*-c)\int_{\gamma_1}\left(\frac{\partial\psi}{\partial z}\right)^2 dz + O(|c^*-c|^2),$$

(3.1)

or

(3) $$\frac{\partial \tau}{\partial c} = 2i \int_{\gamma_1} \left(\frac{\partial \psi}{\partial z}\right)^2 dz \ .^*$$

Let γ^*, γ^{**} denote two piecewise smooth, simple, closed, non-intersecting Jordan curves separating O_1 from $O_2 \cup \beta$. Then by the analyticity of the integrands, the integrations in (2), (3) can be performed along γ^*, γ^{**} instead of γ_1. Thus we find

(4) $$\frac{\partial^2 \tau}{\partial c\, \partial \bar{c}} = 4i \int_{\gamma^*} \frac{\partial^2 \psi}{\partial z\, \partial \bar{c}} \quad \frac{\partial \psi}{\partial z} \quad dz \ ,$$

and from (2) we find for z not on γ^{**} that

(5) $$\frac{\partial^2 \psi}{\partial z\, \partial \bar{c}} = -\ 2i \int_{\gamma^{**}} \frac{\partial^2 G(z,\bar{z},\zeta,\bar{\zeta})}{\partial z\, \partial \bar{\zeta}} \quad \frac{\partial \psi}{\partial \bar{\zeta}} \quad d\bar{\zeta}$$

$$\equiv \pi i \int_{\gamma^{**}} K(z,\bar{\zeta}) \ \frac{\partial \psi}{\partial \bar{\zeta}} \ d\bar{\zeta} \quad .$$

The kernel

$$K(z,\zeta) = -\frac{2}{\pi} \ \frac{\partial^2 G(z,\bar{z},\zeta,\bar{\zeta})}{\partial z\, \partial \bar{\zeta}}$$

was investigated by M. Schiffer and G. Szegö (1949) for the 3-dimensional case.

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* $\frac{\partial \tau}{\partial c}$ is essentially the resultant of the pressure forces acting on γ_1.

(3.1)

For the two-dimensional case see : R. Courant: The Dirichlet-integral (1949), Appendix by M. Schiffer, Ch. 2.2. Since the kernel $K(z, \overline{\zeta})$ is analytic in $\overline{\zeta}$ · $\Delta - O$, even for $z = \zeta$, we may allow γ^{**} to coincide with γ^{*}. Substituting (5) into (4) we find

(6) $$\frac{\partial^2 \tau}{\partial c\, \partial \overline{c}} = -\, 4\pi \int_{\gamma^*} \int_{\gamma^*} K(z, \overline{\zeta})\, \frac{\partial \psi}{\partial z}\, \frac{\partial \psi}{\partial \overline{\zeta}}\, dz\, d\overline{\zeta} \ .$$

We can now remove the restriction that γ_1 be analytic. Let $\{\gamma_{1n}\}$ be a sequence of simple, closed analytic Jordan-curves disjoint from ∂O_2 and of γ^*, and $\gamma_{in} \to \gamma$. Each pair (O_{1n}, O_2) $(\partial O_{1n} = \gamma_{1n})$ defines a $\psi_n(z, \overline{z})$. The sequence $\{\psi_n\}$ approximates ψ uniformly in some range $(c-\delta, c+\delta)$ and in a neighborhood of γ^*. Then $\frac{\partial^2 \psi_n}{\partial z^2} \to \frac{\partial^2 \psi}{\partial z^2}$ uniformly on γ^*, since the functions ψ_n are harmonic. Similarly, with obvious notations

$$K_n(z, \overline{\zeta}) \to K(z, \overline{\zeta})$$

and

$$\tau_n(c, \overline{c}) \to \tau(c, \overline{c})$$

uniformly in some c-neighborhood and on γ^*. Hence by (3) and (6)

$$\frac{\partial \tau_n}{\partial c} \to \frac{\partial \tau}{\partial c}$$

$$\frac{\partial^2 \tau_n}{\partial c\, \partial \overline{c}} \to \frac{\partial^2 \tau}{\partial c\, \partial \overline{c}}$$

and $\frac{\partial \tau}{\partial c}$, $\frac{\partial^2 \tau}{\partial c\, \partial \overline{c}}$ are again given by (3) and (6).

(3.1)

It has been proved by Schiffer and Szegö that the quadratic form defined by K staisfies the inequality

$$\sum_{i,k=1}^{N} K(\zeta_i, \overline{\zeta}_k)\, t_i \overline{t}_k \geq \sum_{i,k=1}^{N} \Gamma(\zeta_i, \overline{\zeta}_k)\, t_i \overline{t}_k$$

$$\equiv \frac{1}{\pi^2} \iint_{\Delta_1} \left| \sum_{i=1}^{N} \frac{t_i}{(z-\zeta_i)^2} \right|^2 dx\, dy$$

for arbitrary N and t_i. From here we find in the limit that

$$(7) \qquad \int_{\gamma^*}\int_{\gamma^*} K(z,\overline{\zeta})\, \frac{\partial\psi}{\partial\overline{\zeta}}\, \frac{\partial\psi}{\partial\overline{\zeta}}\, dz\, d\overline{\zeta} \geq \frac{1}{\pi^2} \iint_{\Delta_1} \left| \int_{\gamma^*} \frac{1}{(z-\zeta)^2}\, \frac{\partial\psi}{\partial\zeta}\, d\zeta \right|^2 dx\, dy \quad .$$

The righthand integral is clearly non-negative. We assert that it cannot vanish. In fact, assume that it vanishes ; then the inner loop-integral would vanish for almost all $z \in \Delta_1$. We find by integration by parts

$$I(z) \equiv \int_{\gamma^*} \frac{1}{(z-\zeta)^2}\, \frac{\partial\psi}{\partial\zeta}\, d\zeta = \int_{\gamma^*} \frac{1}{z-\zeta}\, \frac{\partial^2\psi}{\partial\zeta^2}\, d\zeta \quad .$$

The function I(z) thus suffers a jump $\partial\psi^2/\partial z^2$ upon crossing γ^*, so that $I(z) \equiv 0$ is impossible. Thus inequality (7) implies that the function $\tau(c,\overline{c})$ is strictly superharmonic.

It is an immediate consequence of this, that the set $B \cup \overline{N}$ is connected. If it weren't, say, if

$$B \cup \overline{N} = N_1 \cup N_2 \quad ,$$

N_1, N_2 closed and disjoint, and, say, $\beta \subset N_2$, then by a suitable small translation of N_1 we could reduce $T[\psi_+]$, while keeping $L[\psi_-]$, $A[\psi_-]$ unchanged.

The set $\Gamma = \partial P - \beta'$ is connected for similar reasons.

3.2 The set $O[\psi]$ has no open non-empty subset.

Suppose that $O[\psi]$ contains a circle C: $|z-z_0| < \rho$. We define then the admissible function (III$'$ or IV)

$$\psi^* = \begin{cases} (\omega/4)(|z-z_0|^2 - \rho^2) & \text{in } C \\ \psi(z) & \text{outside } C . \end{cases}$$

Here ω is given for problem IV, and

(1) $$\omega = 2D[\psi] / L[\psi]$$

in case of III$'$. Then

$$T[\psi_+^*] = T[\psi_+] \quad ,$$
$$A[\psi^*] = A[\psi] \quad ,$$
$$L[\psi_-^*] = L[\psi_-] + \frac{\pi}{4}\omega\rho^4$$
$$D[\psi_-^*] = D[\psi_-] + \frac{\pi}{8}\omega^2\rho^4 \quad .$$

Therefore, taking (1) into consideration

(2) $$V'[\psi] = T[\psi_+] - \lambda A[\psi] + \frac{m^2\omega}{2L[\psi_-]}$$

$$(3)\quad \left\{\begin{aligned} V'[\psi^*] &= T[\psi_+] - \lambda A[\psi] + m^2 \frac{D[\psi_-] + (\pi/8)\,\omega^2\rho^4}{\{L[\psi_-] + (\pi/4)\,\omega\rho^4\}^2} \\ &= T[\psi_-] - \lambda A[\psi] + \frac{m^2\omega}{2\,L[\psi_-] + (\pi/2)\omega\rho^4}\ , \end{aligned}\right. \tag{3.3}$$

$$(4)\quad \left\{\begin{aligned} W[\psi^*] &= T[\psi_+] - \lambda A[\psi] + D[\psi_-] - \omega L[\psi_-] - (\pi/4)\omega^2\rho^4 \\ &= W[\psi] - (\pi/4)\,\omega^2\rho^4 \quad . \end{aligned}\right.$$

From (2), (3), (4) follows that

$$V'[\psi^*] < V'[\psi] \ , \quad W[\psi^*] < W[\psi] \ ,$$

a contradiction. Hence $O[\psi]$ contains no circular disk.

<u>3.3 The sets $N[\psi]$ has no "internal" boundary points, i.e.</u>

$$(1) \qquad \partial\overline{N} = \partial N \ .$$

Since the domain N is open, it is the countable union of connected open sets $N_i (i=1,2,\ldots)$. Suppose that say

$$\mathrm{Int}(\overline{N}_1) = N' = N_1 \cup S \ ,$$

S not empty. If ψ is a solution of III′ , let ψ_* denote the solution of the inner minimum-problem III′ for N′ , normed such that

$$(2) \qquad 2D[\psi_*] \,/\, L[\psi_*] = 2D[\psi_-] / L[\psi_-] = \omega \ .$$

If ψ is a solution of IV and ψ_* of the inner problem IV for N′ , (2) is valid by (2.1.1). Then $\psi_* - \psi$ is harmonic, hence by the maximum-principle of harmonic functions

(3.3)

(3) $\psi_*(z) > \psi(z)$ in N' .

Thus

(4) $$\frac{D[\psi_*]}{L[\psi_*]^2} < \frac{D[\psi_-]}{L[\psi_-]^2} \quad , \qquad (III')$$

(5) $$D[\psi_*] - \omega L[\psi_*] < D[\psi_-] - \omega L[\psi_-] \quad . \qquad (IV) .$$

Hence

$$\psi_{**}(z) = \psi_*(z) + \psi_+(z)$$

is admissible III′ or IV, and

$$V'[\psi_{**}] \geq V[\psi] \qquad (III')$$

$$W[\psi_{**}] \geq W[\psi] \qquad (IV)$$

However, since $\psi_{**} = \psi$ outside N_1 , (4) would result in

$$V'[\psi_{**}] < V'[\psi] \ ,$$

and (5) in

$$W[\psi_{**}] < W[\psi] \ ,$$

contrary to the assumption that ψ is the solution of III′ or IV. Hence S is empty, and

$$N_i = \mathrm{Int}(\bar{N}_i)$$

for each i . Consequently

$$N = \bigcup_i N_i = \bigcup_i \mathrm{Int}\, \bar{N}_i \ .$$

(3.4)

By an easy topological consideration

$$\bigcup_i \operatorname{Int} \bar{N}_i = \operatorname{Int}(\bigcup_i \bar{N}_i) = \operatorname{Int} \bar{N} \quad ,$$

hence indeed

$$N = \operatorname{Int}(\bar{N}) \quad .$$

<u>3.4 The set P is simply connected, and the set $N[\psi]$ is the (countable) union of disjoint, open simply connected sets .</u>

<u>Proof. (a) The set $P[\psi]$ is connected.</u> If the flow is a channel flow, then by the equicontinuity property of the members ψ_n of the minimum sequence there is a strip $\pi - \epsilon < \eta < \pi$ which belongs to all $P[\hat{\psi}_n]$ and therefore also to $P[\hat{\psi}]$. If on the other hand the flow is open, then the projections σ_n of the sets $Q[\hat{\psi}_n]$ on the imaginary ζ-axis have by Eq. (2.6.5) uniformly bounded linear measures, and therefore the measure of the projection σ of the set $Q[\hat{\psi}]$ has the same bound. Hence there is a sequence of horizontal lines $\eta = \eta_n$, $\eta_n \to \infty$, belonging to $P[\hat{\psi}]$. If $P[\hat{\psi}]$ is not connected then it can be decomposed in either case into the disjoint, non-empty open sets P_1, P_2 , where P_2 is in some strip $0 < \eta < \eta_o$, and where $\eta_o < \pi$ in case of channel-flows. Along ∂P_2 , $\psi = 0$. If P_2 is bounded, then by the maximum-principle $\psi = 0$ in P_2 , a contradiction. Suppose now that P_2 is unbounded. Then by the maximum-principle

$$\hat{\psi}(\zeta) < \eta < \eta_o$$

in $g(P_2)$, thus $\psi(z)$ is bounded in P_2. Therefore, by the Phragmén-Lindelöf theorem and the maximum principle $\psi(z) = 0$ in P_2, contrary to assumption. Thus P is indeed connected.

(b) The sets P and N_i are simply connected. In fact, if P were multiply connected, then $\Gamma = \partial P - \beta'$ could not be a connected set, contrary to the results of section 3.1.

If some component of N were multiply connected, then this component would surround a set R not belonging to N. Since P is connected, $R \subset O[\psi]$. But then R consists entirely of "internal" boundary points of N: $R \subset \partial\bar{N} - \partial N$, contrary to the result in 3.3, or it contains an open non-empty set, contrary to 3.2.

<u>3.5 Theorem.</u> If $\psi(z)$ is a solution of a minimum-problem III or IV, then

(a) given any $\epsilon > 0$, any point of the set $\Gamma = \partial P - \beta'$ has a neighborhood in P in which

(1) $$|\nabla\psi| > \lambda^{\frac{1}{2}} - \epsilon \quad .$$

(b) in the entire domain P

(2) $$|\nabla\psi| > |g'(z)|(\lambda/\Lambda)^{\frac{1}{2}} \quad .^{*}$$

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* For open flows $\Lambda = 1$ should be substituted.

(3.5)

(c) If the flow is an open flow or θ' is a straight line, then

(3) $$|\nabla\psi| > \lambda^{\frac{1}{2}}$$

holds in the entire domain P.

<u>Proof.</u> It is sufficient to carry out the proof for channel-flows, since open flows can be considered limiting cases of flows in channels such that $\alpha = |f'(\infty)| \to \infty$.

Consider the regular function $z = \mathbf{F}(\theta)$ $(\theta = \varphi + i\psi)$ which maps the strip E into the domain P in such manner that the imaginary axis and the boundary points $\pm\infty$ are preserved. Given the real numbers p and q $(0 < q < \pi)$, let T denote the triangle domain (p-q, p+iq, p+q). We introduce now the function

$$\rho(\theta) = |\varphi - p| + \psi - q$$

which vanishes on the sides (p-q, p+iq) and (p+iq, p+q) of T. Given any $\epsilon > 0$, let τ denote the subset of T where $\psi + \epsilon\,\rho(\theta) < 0$, i.e., the triangle domain $(p - \alpha,\ p + i\alpha\,\varepsilon/(1+\varepsilon),\ p + \alpha)$ and set

$$\psi^{*}(z) = \Psi(\theta) = \begin{cases} \psi + \epsilon\rho(\theta) & \text{in } T - \tau\,, \\ 0 & \text{in } \tau\,, \\ \psi & \text{outside } T\,. \end{cases}$$

$\psi^{*}(z)$ is then continuous in Δ. Let $\psi^{*}(z) = \hat{\psi}^{*}(\zeta)$ by the mapping $z = f(\zeta)$. Then by Eq. (1.2.6)

$$T[\psi^{*}] - T[\psi] = D[\psi^{*}|T] - D[\psi|T],$$

$$(3.5)$$

and after a simple calculation

(4) $$T[\psi^*] - T[\psi] \leq q^2 \varepsilon (1 + C_1 \varepsilon)$$

where C_1 does not depend on q or ε.

Let σ denote the image of τ under the mapping F. Since obviously

(5) $$L[\psi^*] = L[\psi] \quad , \quad A[\psi^*] = A[\psi] + A_\sigma \quad ,$$

we find by (4) and (5)

(6) $$V[\psi^*] - V[\psi] = W[\psi^*] - W[\psi] \leq -\lambda A_\sigma + q^2 \varepsilon (1 + C_1 \varepsilon) \ .$$

If $\psi(z)$ is the solution of III, then

$$V[\psi^*] - V[\psi] \geq 0 \ ,$$

and for IV

$$W[\psi^*] - W[\psi] \geq 0 \ .$$

We find therefore by (6)

(7) $$\lambda \leq \frac{q^2 \varepsilon}{A_\sigma} (1 + C_1 \varepsilon) \ .$$

The area of the τ is

(8) $$A_\tau = q^2 \varepsilon / (1 + \varepsilon) \ .$$

Introducing (8) into (7) yields

(9) $$\lambda \leq (1 + C_2 \varepsilon) A_\tau / A_\sigma \ .$$

We will estimate now the righthand quotient. Since the arithmetic mean is bigger than the geometric mean,

(3.5)

$$\frac{A_\sigma}{A_\tau} = \frac{1}{A_\tau} \iint_\tau |F'(\theta)|^2 d\varphi\, d\psi \geq \exp\left\{ \frac{2}{A_\tau} \iint_\tau \log |F'(\theta)| d\varphi\, d\psi \right\} ,$$

and with (8) and (9)

$$(10) \qquad \frac{1}{q^2 \epsilon} \iint_\tau \log |F'(\theta)|\, d\varphi\, d\psi \leq \log(1/\sqrt{\lambda}) + O(\epsilon) \qquad .$$

The function

$$Z(\theta) = g(F(\theta))$$

maps in a schlicht manner the domain E into $P[\hat{\psi}] \subset E$. We introduce the relationship

$$Z'(\theta) = g'(F(\theta))\, F'(\theta)$$

together with the estimate $|g'(z)| < \Lambda^{\frac{1}{2}}$ into (10). Hence

$$(11) \qquad \left\{ \begin{aligned} & \frac{1}{q^2 \epsilon} \iint_\tau \log |Z'(\theta)|\, d\varphi\, d\psi \\ & \leq \frac{1}{q^2 \epsilon} \iint_\tau \log |g'(F(\theta))|\, d\varphi\, d\psi + \log \lambda^{-\frac{1}{2}} + C_3 \epsilon \\ & \leq \log (\Lambda/\lambda)^{\frac{1}{2}} + C_3 \epsilon \qquad . \end{aligned} \right.$$

$Z(\theta)$ maps $\tilde{\beta} : \psi = \pi$ into $\hat{\beta} : \eta = \pi$. Thus $Z(\theta)$ can be analytically continued into the strip $\pi \leq \psi < 2\pi$ by Schwarz's reflection principle. If $\tau_0(q)$ is the triangle-domain τ with $p = 0$, and $\mathcal{S}(t)$ is the shift operator $\theta \to \theta + t$, then the function

(12) $$u(t) = \frac{1}{q^2\epsilon} \iint\limits_{\mathcal{S}(t)\,\tau_o(q)} \log|Z'(\theta)|\, d\varphi\, d\psi$$

is harmonic in the strip S_ϵ: $0 < \psi < r = 2\pi - q\epsilon/(1+\epsilon)$. We show that u(t) is bounded and continuous in S_ϵ . Eq. (11) already established the boundedness of u(t) on the real axis. Consider the reflection $\tilde{\tau}$ of the triangle $\mathcal{S}(t_1)\tau_o(2q)$ for real t_1 on the line $\tilde{\beta}$. By (11) then

(13) $$\frac{1}{q^2\epsilon} \iint\limits_{\tilde{\tau}} \log|Z'(\theta)|\, d\varphi\, d\psi \le 2\log\frac{\Lambda}{\lambda} + 4C_3\,\epsilon \le C_4 \quad .$$

Observe here that the constant C_4 does not depend on q or ϵ.

The triangle τ_1 the vertices of which bisect the sides of $\tilde{\tau}$, is obtained from $\mathcal{S}(t_1)\,\tau_o(q)$ by a shift $ir = 2\pi i - iq\,\epsilon/(1+\epsilon)$ or from τ_o by a shift $t = t_1 + ir$. Since $\tau_1 \subset \tilde{\tau}$, by (10) therefore

(14) $$u(t) \le C_4$$

if t is on $\tilde{\beta}'' = \{\theta : \psi = r\}$. The same inequality holds if t is real.

Since the mapping accomplished by $Z(\theta)$ is schlicht, Koebe's theorem can be applied after a conformal mapping of the domain E into the unit circle. By routine calculations Koebe's theorem leads to the inequality

(15) $$|Z'(\theta)| \le \frac{e^{2|\varphi|}}{2\sin\psi}\, |Z'(i\pi/2)|$$

in S_ϵ , as $\varphi \to \pm\infty$, and thus by (12)

$$u(\theta) \le 2|\varphi| + C_5$$

By the Phragmén-Lindelöf theorem and the maximum principle therefore (14) holds everywhere in S_ε. The continuity of $u(\theta)$ follows from (15) easily by substitution into (12).

From the existence of an upper bound independent of q and the continuity of $u(t)$ follows the same for

(16) $$v(t) = \frac{1}{q^2\varepsilon} \iint_{S(t)\,\tau_o(q)} \log |F'(\theta)|\, d\varphi\, d\psi .$$

Therefore $G(\theta) = \log |F'(\theta)|$ is also bounded. In fact, if $v(\theta_o) > M$ for some internal point θ_o and real M, then given any $\delta > 0$, q can be chosen so small that $G(\theta) > M-\delta$ in $S(\theta_o)\ \tau_o(q)$, and then also

(17) $$v(\theta_o) > M-\delta .$$

Consider now the function $w(\theta)$ bounded and harmonic in $0 < \psi < \pi/2$ which assumes the boundary values

(18a) $$w(p + i\pi/2) = G(\varphi + i\pi/2) + \log \lambda^{1/2}$$

(18b) $$w(p) = 0$$

The function

(19) $$v^*(\varphi) = \frac{1}{q^2\varepsilon} \iint_{S(t)\,\tau_o(q)} [G(\theta) - w(\theta)]\, d\varphi\, d\psi$$

is harmonic in $0 < \psi < r-\pi/2$, and by (10)

(20) $$v^*(\varphi) \leq \log \lambda^{-1/2} + O(\varepsilon)$$

By (18a) for sufficiently small q

$$v^*(\varphi + i(r-\pi/2)) \leq \log \lambda^{-1/2} + \varepsilon .$$

By the Phragmén-Lindelöf theorem and the maximum-principle follows then that (20) holds in the entire region (φ replaced by θ). Therefore from (19) follows by an argument similar to the one preceding (17) that

(21) $$G(\theta) \leq w(\theta) + \log \lambda^{-1/2} + \varepsilon$$

By the continuity of $w(\theta)$ on the real axis and by $|\nabla \psi| = 1/|F'(\theta)|$ we obtain the statement (1) from (21).

To prove (2), consider again the analytic function

(22) $$Z'(\theta) = g'(F(\theta))\, F'(\theta) \quad .$$

In some vicinity of the lines $\psi = 0$ and $\psi = 2\pi$,

(23) $$\begin{aligned} |Z'(\theta)| &\leq (\lambda^{-\frac{1}{2}} - \varepsilon)\; g'(F(\theta)) \\ &\leq (\lambda^{-\frac{1}{2}} - \varepsilon)\; \Lambda^{\frac{1}{2}} \quad . \end{aligned}$$

Because of (15), we can apply the Phragmén-Lindelöf theorem and then the maximum-principle to $Z'(\theta)$ again and show that (20) holds everywhere in E . Since ε was arbitrary, indeed

$$|Z'(\theta)| \geq (\Lambda/\lambda)^{\frac{1}{2}}$$

for any θ . By (19) then

$$|F'(\theta)| \leq \frac{1}{g'(F(\theta))} \quad (\Lambda/\lambda)^{\frac{1}{2}}$$

resulting after change of variables in (2) .

The proof of (3) : The function $dz/d\theta$ is regular and by (2) bounded in E. Therefore in case of open flows it follows from (1) by the Phragmén-Lindelöf theorem and the maximum-principle that

$$\left| \frac{dz}{d\theta} \right| < \lambda^{-\frac{1}{2}} \quad (\theta \in E)$$

(3.6)

which implies (3). In case of channelflows with straight β', the function $\psi(z)$ can be continued into the mirror image of Δ on the line β' by Schwarz's reflection principle, and then the same reasoning applies as for open flows.

The theorem just proved is quite fundamental in the investigation of the properties of the solution. In fact, the remainder of Part III of this treatment contains essentially nothing but corollaries of this theorem.

3.6 The boundary of N contains an arc of β of positive length.* For, by the theorem 3.1 and the symmetrization there is a segment $I = (0, i\delta)$ $(\delta > 0)$ of the imaginary axis in N. (At the normalization of $f(\zeta)$, $f(0) = 0$ was agreed upon.)

The solution of the restricted outer minimum-problem can be only increased by a decrease of the set N because of the maximum-principle. Hence

(1) $$\psi_+(z) \leq \psi^*(z)$$

where $\psi^*(z)$ is the streamfunction of the flow in Δ over I (i.e., for which $O[\psi^*] = I$). Since

$$\left|\nabla\psi^*\right|_{z=0} = 0 \text{ , also } \left|\nabla\psi\right|_{z=0} = 0 \quad .$$

By the theorem 3.5 therefore the point $z = 0$ does not belong to ∂P; consequently a circle $C : |z| < \rho$ exists such that $C \cap \Delta \subset N$.

---

* Except if the set N is empty. (Trivial solution, see next section).

3.7 Trivial solutions.

We will say that if

$$\psi_o(z) = \operatorname{Im} g(z)$$

is a solution of a minimum-problem, it is the trivial solution.*

The solution of III is never trivial if a positive m is specified. The question arises whether the solutions of IV are not trivial ?

A sufficient criterion in light of theorem 3.5 is that if anywhere in Δ

$$|\nabla\psi_o(z)| < \lambda^{\frac{1}{2}} \quad ,$$

then ψ_o cannot be a solution of IV. With the usual notation

$$M = 1/|f'(0)|^2$$

IV will have a non-trivial solution for any (λ, ω), for which IV is well posed, and

$$(1) \qquad M < \lambda \ .$$

By definition $M \leq \Lambda$. If $\Delta \neq E$, then $M < \Lambda$, and (1) is satisfied in the strip $M < \lambda < \Lambda$ of the (ω, λ) plane.

Goldshtik (1962) proved (although for bounded domains and different boundary conditions) that for $\lambda = 0$, the differential equations have no other than trivial solutions below some value of ω. This is in harmony with the

---

* In other words, the trivial solution is the streamfunction of a potential flow in Δ.

present findings.

<u>3.8 The set $\Gamma = \partial P - \beta'$ is a rectifiable curve.</u> In fact, let $u, v \in \Gamma$. The function $w = F(z)$ maps P into E conformally, and let $z = G(w)$ denote its inverse. The image of u can be defined as the closed set

$$F(u) = \bigcap_{\Sigma \in S_u} \overline{F(\Sigma)}$$

where S_u is a basis belonging to u in the topology of P. F(v) is defined similarly.

Let the interval (ρ, σ) denote the convex hull of the set $F(u) \cup F(v)$, with $\rho \in F(u)$, $\sigma \in F(u)$ real, and let $\rho_n = \rho + i/n$, $\sigma_n = \sigma + i/n$. Then $G(\rho_n) \to u$, $G(\sigma_n) \to v$. The intervals $J_n = (\rho_n, \sigma_n)$ are mapped by G into the curves Θ_n. If ℓ_n is the length of Θ_n, then from (3.5.1) follows that given $\epsilon > 0$, for sufficiently large n

$$(1) \qquad \ell_n = \int_{\rho_n}^{\sigma_n} \left|\frac{d\theta}{dz}\right|^{-1} |d\theta| \leq \lambda^{-\frac{1}{2}} (\sigma - \rho) + \epsilon .$$

Thus Θ_n are represented by vectors $z = Z_n(\varphi)$ of the uniformly bounded variation (1). By Helly's theorem of choice we can therefore select a subsequence $Z_{n_k}(\varphi)$ converging to some $Z_\infty(\varphi)$ of total variation $\lambda^{-\frac{1}{2}}(\sigma - \zeta)$. $Z_\infty(\varphi)$ represents therefore a ractifiable curve X_∞. The curves Θ_n are subsets of the level-curves $\psi = 1/n$, so that the curve X_∞ must be a subset of ∂P, and must connect u and v.

3.9 The set $P[\psi]$ has no "internal" boundary points, i.e.

(1) $$\partial\bar{P} = \partial P \ .$$

We have to show that $\mathrm{Int}(\bar{P}) = P$. Clearly

$$P \subset \mathrm{Int}(\bar{P}) = P'$$

Suppose that

(2) $$P' = P \cup S,$$

where S is not empty. Then $S \subset \partial P$, thus

$$\psi(z) = 0 \quad \text{if} \quad z \in S \ .$$

We define the function $\psi_+^*(z)$ such that the restricted outer minimum-problem defined in the domain P' has a solution $\psi_+^*(z)$, $\psi_+^*(z) > 0$ in P', hence also on S. Therefore by the maximum-principle

$$\psi_+^*(z) > \psi(z) \qquad \text{in } P \ .$$

Hence by the definition of ψ_+^*,

(3) $$T[\psi_+^*] < T[\psi_+] \ .$$

The function

(4) $$\psi^*(z) = \psi_+^*(z) + \psi_-(z)$$

is a competing function for III, IV respectively. The equation

$$L[\psi_-^*] = L[\psi_-]$$

is obvious ;

$$A[\psi^*] = A[\psi]$$

$$(3.10)$$

is valid, because the sets $Q[\psi^*]$, $Q[\psi]$ differ only in a subset of Γ which is a rectifiable curve and therefore of zero plane measure. Hence

$$V[\psi^*_{III}] \geq V[\psi_{III}]$$

and

$$W[\psi^*_{IV}] \geq W[\psi_{IV}] \quad ,$$

with obvious indexing of ψ. However, from (3), (4)

$$V[\psi^*] = T[\psi^*_+] + D[\psi_-] - \lambda A[\psi^*] < T[\psi_+] + D[\psi_-] - \lambda A[\psi] = V[\psi] \quad ,$$

and similarly

$$W[\psi^*] < W[\psi] \; ,$$

a contradiction. Hence S defined in (2) is empty.

3.10 $\partial P \cap \Delta = \partial N \cap \Delta = O[\psi] = \gamma$.

The relationships

$$\partial P \cap \Delta \subset O \; , \quad \partial N \cap \Delta \subset O$$

are consequences of the continuity of $\psi(z)$. Therefore it is sufficient to show that

(1) $$O[\psi] \subset \partial P \cap \partial N \equiv \gamma \; .$$

By (3.3.1)

$$\partial N = \partial \overline{N} = \partial(\Delta - P) \subset B \cup \partial P \quad ,$$

hence

(2) $$\partial N \cap \Delta \subset \partial P \cap \Delta \quad .$$

Similarly, by (3.9.1)

$$\partial P = \partial \overline{P} = \partial(\Delta - N) \subset B \cup B' \cup \partial N \; ,$$

(3.10)

hence considering also (2) ,

(3) $$\partial P \cap \Delta = \partial N \cap \Delta \, .$$

Since $O[\psi]$ contains no open non-empty set,

(4) $$O[\psi] \subset (\partial P \cup \partial N) \cap \Delta \quad ,$$

therefore Eqs. (3), (4) result in (1) .

PART IV

INTEGRAL EQUATION AND APPLICATIONS

4.1 The method of interior variations. Garabedian and Spencer (1952) showed how restricted "analytic" variations of the domain can be applied to deduce from a minimum-principle boundary conditions in explicit form, even though smoothness of the domain boundary is not a priori known. In the problems of free boundary problems discussed by them even the analiticity of the unknown boundary curve can be deduced. In the present problem the boundary curve is not a free boundary, it only separates two domains in which the solution satisfies different differential equations. It is therefore questionable whether the boundary is analytic, and there seems no known method available to prove it. Nevertheless, the method of interior variations will be quite useful.

We will apply a variant of the methods of Garabedian-Spencer (1952), Garabedian-Lewy-Schiffer (1952) and Garabedian (1964), Chapter 15. The interior variations will be given in the halfplane or strip domain E connected with the flow domain Δ by the conformal mapping functions $z = f(\zeta)$, $\zeta = g(z)$, (rather than in Δ itself). It will be assumed that $f(\zeta)$ satisfies the conditions in Section 1.1.

(4.1)

Let ϵ_o denote a small positive number and $F(\zeta, \bar{\zeta})$ a complex valued function with finite Dirichlet-integral. Then we set

(1a) $$\zeta^* = \zeta + \epsilon_o \, F(\zeta, \bar{\zeta})$$

(1b) $$\hat{\psi}^* (\zeta, \bar{\zeta}) = \hat{\psi}(\zeta^*, \bar{\zeta}^*) \quad {}^*$$

ϵ_o should be chosen as small that (1a) is a schlicht mapping, F should be such that the boundary line(s) $\eta = 0$, $(\eta = \pi)$ be mapped into themselves. We will also assume that F together with its first and second derivatives is bounded.

We proceed to estimate the variation $T[\hat{\psi}^*] - T[\hat{\psi}]$. Let Σ_r denote the preimage of the domain $\Sigma_r^* = \{ \zeta^* : |\zeta^*| < r, \ \zeta^* \in E \}$ under the mapping (1a). Then by Eq. (1.2.5)

$$T[\hat{\psi}^*] = \lim_{r \to \infty} \left\{ \iint_{\Sigma_r} \left(\left| \frac{\partial \hat{\psi}^*}{\partial \zeta} \right|^2 + 1 \right) d\xi \, d\eta + 2 \, \mathrm{Re} \int_{\partial \Sigma_r} \hat{\psi}^* \, d\zeta \right\}$$

(2)

$$\equiv \lim (T_1 + T_2) \quad .$$

We use the estimate

(3) $$\frac{\partial(\xi, \eta)}{\partial(\xi^*, \eta^*)} = \left(1 - 2 \, \epsilon_o \, \mathrm{Re} \, \frac{\partial F}{\partial \zeta} \right) + \theta . K . \, \epsilon_o^2$$

where $|\theta| < 1$ and K is independent of r. Thus we find by routine calculations from (2) and (3)

---

* For the sake of clarity here and in later sections $h(\zeta)$ will denote an analytic function of ζ, $k(\zeta, \bar{\zeta})$ any function depending on ξ and η.

(4.1)

$$
(4)\quad \begin{cases} T[\psi^*] - T[\psi] = \epsilon_o \lim\limits_{r\to\infty} \operatorname{Re}\left\{ 8 \iint\limits_{\Sigma_r^*} \left(\dfrac{\partial\hat{\psi}}{\partial\zeta}\right)^2 \dfrac{\partial F}{\partial\bar{\zeta}}\, d\xi\, d\eta \right. \\[2ex] \left. - \int\limits_{\partial\Sigma_r^*} \left(2\hat{\psi} - \dfrac{i}{2}\bar{\zeta}\right) dF(\zeta,\bar{\zeta}) + \dfrac{i}{2}\int\limits_{\partial\Sigma_r^*} \bar{F}\, d\zeta \right\} + O(\epsilon_o^2) . \end{cases}
$$

We will compute next the variation of the integral

$$
I = \iint\limits_{\psi<0} U(z,\bar{z})\, dx\, dy = \iint\limits_{\hat{\psi}(\zeta)<0} \hat{U}(\zeta,\bar{\zeta})\, |f'(\zeta)|^2\, d\xi\, d\eta ,
$$

where $\hat{U}(\zeta,\bar{\zeta}) = U(z,\bar{z}) = 2\omega\hat{\psi}(\zeta,\bar{\zeta}) - \lambda$. Then

$$
(5)\quad I^* = \iint\limits_{\psi^*(z)<0} U^*(z,\bar{z})\, dx\, dy = \iint\limits_{\hat{\psi}^*(\zeta)<0} \hat{U}^*(\zeta,\bar{\zeta})\, |f'(\zeta)|^2 d\xi\, d\eta
$$

$$
= \iint\limits_{\hat{\psi}^*(\zeta)<0} \hat{U}^*(\zeta,\bar{\zeta})\, |f'(\zeta)|^2\, \frac{\partial(\xi,\eta)}{\partial(\xi^*,\eta^*)}\, d\xi^*\, d\eta^*
$$

where

$$
\hat{U}^*(\zeta,\bar{\zeta}) = \hat{U}(\zeta^*,\bar{\zeta}^*) , \quad \hat{\psi}^*(\zeta,\bar{\zeta}) = \hat{\psi}(\zeta^*,\bar{\zeta}^*) .
$$

With the variational formulas (1) then

$$
|f'(\zeta)|^2 = |f'(\zeta^*)|^2 - 2\epsilon_o \operatorname{Re}\left\{\overline{f'(\zeta^*)}\, f''(\zeta^*)\, F(\zeta^*,\bar{\zeta}^*)\right\} + O(\epsilon_o^2) .
$$

Substituting this and (3) into (5) leads to

$$
(6)\quad I^* - I = -2\epsilon_o \operatorname{Re}\left\{ \iint\limits_{\hat{N}} \left[2\omega\hat{\psi}(\zeta,\bar{\zeta}) - \lambda\right] |f'(\zeta)|^2 \right.
$$

$$
\left. \times\left[F_\zeta(\zeta,\bar{\zeta}) + \frac{f''(\zeta)}{f'(\zeta)}\, F(\zeta,\bar{\zeta})\right] d\xi\, d\eta \right\} + O(\epsilon_o^2) .
$$

(4.1)

We specify now the function $F(\zeta, \bar{\zeta})$ <u>for open flows.</u> Let τ denote any point in E, and ρ any positive number for which the circle $C_\rho = \{\zeta : |\zeta - \tau| < \rho\}$ is entirely in E. We set

$$(7) \qquad F(\zeta) = \begin{cases} e^{i\alpha}(\bar{\zeta} - \bar{\tau})/\rho^2 + e^{-i\alpha}/(\zeta - \bar{\tau}) & \text{if } \zeta \in C_\rho \\ e^{i\alpha}/(\zeta - \tau) + e^{-i\alpha}/(\zeta - \bar{\tau}) & \text{if } \zeta \in E - C_\rho \, . \end{cases}$$

α is here an arbitrary real number .

It is easy to see that for sufficiently small values of ϵ_o , $\zeta^* = \zeta + \epsilon_o F(\zeta, \bar{\zeta})$ is a schlicht mapping of E onto itself. We introduce now (7) into (4) and (6). A certain care is required in the evaluation of the line integrals in (4) . Let us denote $\zeta = re^{i\theta}$, $\tilde{\psi}(r, \theta) = \hat{\psi}(\zeta, \bar{\zeta})$,

$$J(r) = \int_0^\pi \left[\tilde{\psi}(r, \theta) - r \sin\theta \right] d\theta \quad .$$

If

$$\Omega_{ab} = \left\{ \zeta : a < |\zeta| < b, \ \eta > 0 \right\}$$

$$\Gamma_r = \left\{ \zeta : |\zeta| = r , \quad \eta > 0 \right\}$$

then by Schwarz's inequality

$$|J(b) - J(a)|^2 = \left[\iint_{\Omega_{ab}} \left(\frac{\partial \tilde{\psi}}{\partial r} - \sin\theta \right) dr\, d\theta \right]^2$$

$$\leq \iint_{\Omega_{ab}} \left[\frac{\partial}{\partial r} (\tilde{\psi} - r \sin\theta) \right]^2 r\, dr\, d\theta \iint_{\Omega_{ab}} \frac{dr}{r} \quad d\theta$$

$$\leq \pi D\left[\hat{\psi} - \eta \mid \Omega_{ab} \right] \log(b/a) \quad .$$

(4.1)

Keeping a fixed, and letting $b \to \infty$, we find by the boundedness of the right hand Dirichlet-integral for admissible functions $\hat{\psi}$ that for $b \to \infty$,

(8) $$|J(b)| = O(\sqrt{\log b}) \quad .$$

If $|\zeta| > 2|\tau|$, then by (7) a constant C_1 exists, such that

$$\left|\frac{\partial F}{\partial \theta}\right| < \frac{C_1}{r} \quad .$$

Therefore by (8)

$$\left| \int_{\Gamma_r} (\hat{\psi} - \eta) \frac{\partial F}{\partial \theta} d\theta \right| = O(r^{-1} \sqrt{\log r})$$

and thus

$$\lim_{r \to \infty} \int_{\Gamma_r} \hat{\psi} \frac{\partial F}{\partial \theta} d\theta = \lim_{r \to \infty} \int_{\Gamma_r} r \sin\theta \frac{\partial F}{\partial \theta} d\theta = -\pi \cos\alpha \quad .$$

Evaluation of the other terms in (4) is straightforward. Thus we find for small ρ

(9) $$\begin{cases} T[\psi^*] - T[\psi] = \epsilon_o \operatorname{Re}\left\{ \dfrac{8e^{i\alpha}}{\rho^2} \displaystyle\iint_{C_\rho} \left(\frac{\partial \hat{\psi}}{\partial \zeta}\right)^2 d\xi\, d\eta + 2\pi e^{i\alpha} \right\} + O(\epsilon_o^2) \\ \qquad = 2\pi \operatorname{Re}\left\{ 4\epsilon \hat{\psi}_\tau(\tau, \bar{\tau})^2 + \epsilon \right\} + O(\rho|\epsilon|) + O(|\epsilon|^2) \end{cases}$$

where $\epsilon = \epsilon_o e^{i\alpha}$. Similarly from (6)

(10) $$\begin{cases} I^* - I = 2\operatorname{Re}\left\{ \epsilon \displaystyle\iint_{\hat{\psi} < 0} \left[2r\hat{\psi}(\zeta, \bar{\zeta}) - \lambda \right] \left[\frac{\kappa(\zeta, \bar{\zeta})}{(\zeta - \tau)^2} + \frac{1}{(\bar{\zeta} - \tau)^2} \right. \right. \\ \qquad \left. \left. - \dfrac{f''(\zeta)}{f'(\zeta)} \left(\dfrac{\kappa(\zeta, \bar{\zeta})}{\zeta - \tau} + \dfrac{1}{\bar{\zeta} - \tau} \right) \right] |f'(\zeta)|^2 \, d\xi\, d\eta \right\} + O(|\epsilon|^2) \end{cases}$$

(4.1)

where $\nu(\zeta, \overline{\zeta})$ is the characteristic function of the set $E - C_\rho$.

Suppose first that $\psi(z, \overline{z})$ is a solution of minimum-problem IV. Then it is also a solution of the minimum-problem obtained by narrowing down the class of admissible functions to those obtained from $\psi(z, \overline{z})$ by the interior variations of the of the domain Δ specified in Eqs. (1) and (7), say $\Psi(z, \overline{z}; \epsilon)$. (Then of course, $\psi(z, \overline{z}) = \Psi(z, \overline{z}; 0)$.) The functionals $T[\Psi]$, $I[\Psi]$ become now ordinary functions of the complex variable ϵ . Hence for $\Psi(z, \overline{z}; \epsilon)$ to be a solution of the minimum-problem IV, we must have

$$(11) \quad \frac{\partial}{\partial \epsilon} \left\{ T[\Psi] - \omega L[\Psi_-] - \lambda A[\Psi] \right\}_{\epsilon = 0} = \frac{\partial}{\partial \epsilon} \left\{ T[\Psi] + I[\Psi] \right\}_{\epsilon = 0} = 0 .$$

If now ψ is a solution of minimum-problem III , then (11) is still valid, but ω is then an unknown Lagrange multiplier , to be chosen such that $L[\Psi_-] = m$ can be realized. The derivative (11) is obtained immediately from Eqs. (9), (10) leading to

$$(12) \left\{ \begin{aligned} 4\left(\frac{\partial \psi(\tau, \overline{\tau})}{\partial \tau}\right)^2 + 1 = -\frac{1}{\pi} \iint_{\hat{\psi} < 0} & \left[2\omega\hat{\psi}(\zeta, \overline{\zeta}) - \lambda\right] \left\{ \frac{\varkappa(\zeta, \overline{\zeta})}{(\zeta - \tau)^2} + \frac{1}{(\overline{\zeta} - \tau)^2} \right. \\ & \left. - \frac{f''(\zeta)}{f'(\zeta)} \left[\frac{\varkappa(\zeta, \overline{\zeta})}{\zeta - \tau} + \frac{1}{\zeta - \tau}\right] \right\} |f'(\zeta)|^2 \, d\xi \, d\eta + O(\rho) . \end{aligned} \right.$$

The left hand side is independent of ρ . Thus letting $\rho \to 0$, we find

$$(13) \quad 4\left(\frac{\partial \hat{\psi}(\tau, \overline{\tau})}{\partial \tau}\right)^2 + 1 = -\frac{1}{\pi} \iint_{\hat{\psi} < 0} \left[2\omega\hat{\psi}(\zeta, \overline{\zeta}) - \lambda\right] \left[\hat{K}(\zeta, \tau) + \overline{\hat{K}(\zeta, \overline{\tau})}\right] |f'(\zeta)|^2 \, d\xi \, d\eta ,$$

(4.1)

where
$$\hat{K}(\zeta,\tau) = \frac{1}{(\zeta-\tau)^2} - \frac{f''(\zeta)}{f'(\zeta)}\,\frac{1}{\zeta-\tau} .$$

We return now to the "physical" coordinates x, y, $z = x + iy$:

$$(14)\quad 4\left(\frac{\partial\psi(t,\bar{t})}{\partial t}\right)^2 = -\frac{1}{\pi}\iint\limits_{\psi<0}\left[2\mu\psi(z,\bar{z}) - \lambda\right]\left[K(z,t) + \tilde{K}(\bar{z},t)\right]dx\,dy - g'(t)^2 ,$$

where

$$(15/o)\quad K(z,t) = g'(t)^2\left\{\frac{1}{[g(z)-g(t)]^2} + \frac{g''(z)}{g'(z)^3}\,\frac{1}{g(z)-g(t)}\right\} ,$$

$$(16/o)\quad \tilde{K}(z,t) = g'(t)^2\left\{\frac{1}{[\bar{g}(z)-g(t)]^2} + \frac{\bar{g}''(z)}{\bar{g}'(z)^3}\,\frac{1}{\bar{g}(z)-g(t)}\right\} .$$

The function $\bar{g}(z)$ here is defined by the relationship

$$\bar{g}(z) = \overline{g(\bar{z})}$$

and is analytic in the reflection of Δ to the real axis.

For channelflows similar results can be obtained if we define

$$C_\rho = \left\{\zeta : |\tanh(\zeta-\tau)/2| < \rho\right\} ,$$

$$F(\zeta,\bar{\zeta}) = \begin{cases} e^{i\alpha}\tanh\dfrac{\bar{\zeta}-\bar{\tau}}{2} / \rho^2 + e^{-i\alpha}\coth\dfrac{\zeta-\bar{\tau}}{2} & \text{if } \zeta\in C_\rho , \\ e^{i\alpha}\coth\dfrac{\zeta-\tau}{2} + e^{-i\alpha}\coth\dfrac{\zeta-\bar{\tau}}{2} & \text{if } \zeta\in E - C_\rho . \end{cases}$$

Then using the same technique as for open flows we find that (14) is again valid with

(4.2)

$$(15/ch) \quad K(z,t) = \frac{g'(t)^2}{\sinh[g(z)-g(t)]}\left\{\coth[g(z)-g(t)] + \frac{g''(z)}{g'(z)^3}\right\},$$

and

$$(16/ch) \quad \tilde{K}(z,t) = \frac{g'(t)^2}{\sinh[\overline{g}(z)-g(t)]}\left\{\coth[\overline{g}(z)-g(t)] + \frac{\overline{g}''(z)}{\overline{g}'(z)^3}\right\}.$$

The functions $\tilde{K}(z,t)$ defined by (16/o) and (16/ch) are analytic if $t\in\Delta$ and $z\in\tilde{\Delta}$, where $\tilde{\Delta}$ is the mirror image of Δ to the real axis. The function $K(z,t)$ can be written in both cases in the form

$$(17) \qquad K(z,t) = \frac{1}{(z-t)^2} + k(z,t)$$

where $k(z,t)$ is analytic in both variables if $z\in\Delta$, $t\in\Delta$.

4.2. The matching condition. We want to use now the integral equation (4.1.14) to derive a matching condition on the boundary γ between P and N. The set N is in general an open set consisting of finite or countably many disjoint open connected sets. Let N' denote one of them. For the derivation of the matching condition it suffices to investigate the integral equation in any neighborhood of the curve γ. Therefore we write it in the form

$$(1) \quad 4\pi\psi_t(t,\overline{t})^2 = -\iint_{N'} [2\omega\psi(z,\overline{z}) - \lambda]\frac{1}{(z-t)^2}\,dxdy + A'(t)$$

where $A'(t)$ is an analytic function in $\mathrm{Int}(\overline{P\cup N'})$. We will show first that the term

$$J(t,\overline{t}) = \iint_{N'} \frac{\psi(z,\overline{z})}{(z-t)^2}\,dxdy$$

(4.2)

is uniformly bounded. We obtain by integration by parts, taking into consideration that ψ vanishes on $\partial N'$,

(2) $$J(t,\bar{t}) = \iint_{N'} \frac{\psi_z(z,\bar{z})}{z-t} \, dx\,dy \; .$$

Let N_ε denote the open set where $\psi < -\varepsilon$, $N_\varepsilon \subset N'$. Then by integration by parts

(3) $$\begin{cases} J_\varepsilon(t,\bar{t}) = \displaystyle\iint_{N_\varepsilon} \frac{\psi_z(z,\bar{z})}{z-t} \, dx\,dy = \frac{i}{2} \int_{\partial N_\varepsilon} \frac{\bar{z}-\bar{t}}{z-t} \, \psi_z(z,\bar{z})\, dz - \iint_{N_\varepsilon} \frac{\bar{z}-\bar{t}}{z-t} \, \psi_{z\bar{z}} \, dx\,dy \\[2ex] \qquad = \displaystyle\frac{i}{2} \int_{\partial N_\varepsilon} \frac{\bar{z}-\bar{t}}{z-t} \, \psi_z(z,\bar{z}) \, dz - \frac{\omega}{4} \iint_{N_\varepsilon} \frac{\bar{z}-\bar{t}}{z-t} \, dx\,dy \; , \end{cases}$$

where in the second equation $\psi_{z\bar{z}} = \omega/4$ was taken into account. From $\Delta\psi = \omega$ follows by Gauss' identity that

(4) $$\iint_{\partial N_\varepsilon} \frac{\partial\psi}{\partial n} \, ds = \omega A_{N_\varepsilon} \; .$$

$\partial\psi/\partial n > 0$ almost everywhere on ∂N_ε because of (2.5.12). Hence (4) can be written in the form

$$2 \int_{\partial N_\varepsilon} \left|\frac{\partial\psi}{\partial z}\right| |dz| = \omega A_{N_\varepsilon} < \omega A_{N'} \leq \omega A_N \; .$$

Therefore from (3) we obtain the preliminary estimate

$$(5) \qquad |J_\varepsilon(t,\bar{t})| = \left| \iint_{N_\varepsilon} \frac{\psi_z(z,\bar{z})}{z-t}\, dx\, dy \right| \leq \frac{1}{2}\, \omega A_N \quad .$$

Since this inequality is valid for all $\varepsilon > 0$, even

$$(6) \qquad J(t,\bar{t}) < \frac{1}{2}\, \omega A_N \quad .$$

Next we estimate the integral

$$(7) \qquad J^*(t,\bar{t}) = \iint_{N'} \frac{dx\, dy}{(z-t)^2} \quad .$$

Let N'' denote the intersection of N' with the circle $|z-t| < 1$. Then by integration by parts

$$J^*(t,\bar{t}) = \frac{i}{2} \int_{\partial N''} \frac{d\bar{z}}{z-t} + \iint_{N-N''} \frac{dx\, dy}{(z-t)^2} \quad .$$

If d is the distance of t from $\partial N'$ and $|t| < R$, then from here

$$(8) \qquad |J^*(t,\bar{t})| \leq \frac{1}{2d} \int_{\partial N''} |dz| + A_{N'} \leq \frac{C_1}{d} \quad ,$$

where C_1 may depend on R but is otherwise independent of t.

Substitution of (6) and (8) into (1) yields

$$(9) \qquad |\psi_t(t,\bar{t})| \leq c\, d^{-\frac{1}{2}} \quad ,$$

implying also

$$(10) \qquad |\psi(t,\bar{t})| \leq 4\, c\, d^{\frac{1}{2}} \quad .$$

(4.2)

Therefore ψ itself satisfies a Hölder condition with exponent $\frac{1}{2}$, in any subset M_R: $|t| < R$ of N'. This fact implies however (cf. Courant-Hilbert II, p. 353) that

$$I_R \equiv \iint_{M_R} \frac{\psi(z,\bar{z})-\psi(t,\bar{t})}{(z-t)^2}\,dx\,dy$$

is also Hölder - continuous in M_R. The remainder $I_\infty - I_R$ is for sufficiently large R analytic there. Thus (1) can be written in the form

$$(11) \qquad 4\pi\,\psi_t(t,\bar{t})^2 = -\left[2\omega\,\Theta(t,\bar{t})\,\psi(t,\bar{t})-\lambda\right]\iint_{N'} \frac{dx\,dy}{(z-t)^2} + C(t,\bar{t})$$

where $C(t,\bar{t})$ is continuous in Δ, and $\Theta(t,\bar{t})$ is the characteristic function of N'.

We will transform now the terms of (1) into the ζ-plane in order to make use of the symmetrization properties of $\hat{\psi}(\zeta)$. The integral $J^*(t,\bar{t})$ (Eq. (7)) can be transformed as follows.

$$(12) \qquad \begin{cases} J^*(t) = \displaystyle\iint_{\hat{N}'} \frac{|f'(\zeta)|^2}{[f(\zeta)-f(\tau)]^2}\,d\xi\,d\eta \\[2ex] \quad = \displaystyle\frac{\overline{f'(\tau)}}{f'(\tau)}\iint_{\hat{N}'} \frac{d\xi\,d\eta}{(\zeta-\tau)^2} + \frac{\overline{f''(\tau)}}{f'(\tau)}\iint_{\hat{N}'} \frac{\bar{\zeta}-\bar{\tau}}{(\zeta-\tau)^2}\,d\xi\,d\eta + A(\tau) \end{cases}$$

where $A(\tau)$ is continuous in E. By integration by parts

$$\iint_{\hat{N}'} \frac{\bar{\zeta}-\bar{\tau}}{(\zeta-\tau)^2}\,d\xi\,d\eta = \frac{i}{2}\int_{\partial\hat{N}'} \frac{\bar{\zeta}-\bar{\tau}}{\zeta-\tau}\,d\bar{\zeta}$$

hence the second right hand term in (12) is bounded. Let us examine

$$\iint_{\hat{N}'} \frac{d\xi d\eta}{(\zeta-\tau)^2} = \frac{i}{2} \int_{\partial\hat{N}'} \frac{d\zeta}{\zeta-\tau} .$$

Let τ_o denote a point of $\partial N'$ in which the latter has a tangent, with unit vector $e^{i\alpha}$. Given $\epsilon > 0$ and $0 < \beta_o < \pi/2$, let

(13) $$\tau_\epsilon = \tau_o + \epsilon e^{i(\alpha+\beta-\pi/2)}$$

where $\beta = \beta(\epsilon)$, $|\beta| < \beta_o$*. Let Ω_ϵ denote the path $\partial\hat{N}' = \Omega'$ without the arc $|\mathrm{Im}\,\tau_o - \eta| < \epsilon$. Then by Privaloff's lemma (cf Privaloff (1956) Ch. 3, § 2.)

(14) $$\lim_{\epsilon\to 0} \left\{ \int_{\Omega'} \frac{d\zeta}{\zeta-\tau_\epsilon} - \int_{\Omega_\epsilon} \frac{d\zeta}{\zeta-\tau_o} \right\} = \pi i \left[\dot{\tau}^2\right]_{\tau=\tau_o} ,$$

the dot denoting differentiation with respect to the arc length.

We combine with (14) the limit relationship

$$\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} \frac{d\zeta}{\zeta-\tau_o} = \pi i ,$$

valid for any τ_o in which Ω' has a tangent. Thus

$$\lim_{\epsilon\to 0} \left\{ \int_{\Omega'} \frac{d\zeta}{\zeta-\tau_\epsilon} - 2i \int_{\Omega_\epsilon} \frac{d\eta}{\zeta-\tau_o} \right\} = \pi i \left\{ \left[\dot{\tau}^2\right]_{\tau_o} + 1 \right\} ,$$

---

* We will say that $\tau_\epsilon \to \tau_o$ "non-tangentially" if $\epsilon \to 0$. Notation: $\tau_\epsilon \not\to \tau_o$.

(4.2)

i.e., for $\varepsilon \to 0$

$$\int_{\Omega} \frac{d\bar{\zeta}}{\zeta - \tau_\varepsilon} = 2i \int_{\Omega_\varepsilon} \frac{d\eta}{\zeta - \tau_o} + O(1) .$$

η is a monotonic function of the arclength along the curve Ω' because of the symmetrization property of $\hat{\psi}(\zeta, \bar{\zeta})$. Therefore

$$\left| \int_{\Omega'_\varepsilon} \frac{d\eta}{\zeta - \tau_o} \right| \leq \int_{\Omega_\varepsilon} \frac{d\eta}{|\eta - \eta_o|} = 2 \log \frac{1}{\varepsilon} + C',$$

or

$$\left| \int_{\Omega'} \frac{d\bar{\zeta}}{\zeta - \tau_\varepsilon} \right| \leq 4 \log \frac{C}{\varepsilon} . \tag{15}$$

By (11) therefore constants C_1, C_2 exist, such that

$$J^*(t) \leq C_1 \log (C_2/\varepsilon) , \tag{16}$$

where

$$t = f\left(\tau_o + \varepsilon e^{i(\hat{\alpha} + \beta)}\right) , \tag{17}$$

and $e^{i\hat{\alpha}}$ is the tangent unit-vector of $\hat{\gamma}$ in τ_o.

By a theorem of Privaloff (1919) the mapping $f(\tau)$ is angle-preserving in the points of $\hat{\gamma}$ with the exception of a set of measure zero. By the theorem of F. and M. Riesz (1916) a set of measure zero of $\hat{\gamma}$ is mapped by $f(\tau)$ into a measure zero of γ. Therefore non-tangential approach to $\hat{\gamma}$ is equivalent to non-tangential approach to γ. Hence the appraisal (16) remains valid with

(4.2)

suitably chosen constants C_1, C_2, even if (17) is replaced by

$$z = t + \varepsilon e^{i(\alpha+\beta)}$$

where $t \in \gamma$, and $e^{i\alpha}$ is the tangent unitvector of γ in the point t. Taking (10) into consideration, we find from (16) that

$$\psi(t) \iint_{N'} \frac{dxdy}{(z-t)^2} \to 0 \tag{18}$$

if $z \to t$ non-tangentially, for almost all $t \in \gamma'$. Let now

$$z_P = t - \varepsilon e^{i(\alpha+\beta+\pi/2)}, \; z_N = t + \varepsilon e^{i(\alpha+\beta+\pi/2)} .$$

For sufficiently small ε the segment $\overline{t_o t_1} \in P$, $\overline{t_o t_2} \in N$ for almost all $t \in \gamma'$. By the theorem of Golubew* (cf. Privalow (1956))

$$\lim_{\varepsilon \to 0} \left\{ \int_{\partial N'} \frac{d\bar{z}}{z-t_P} - \int_{\partial N'} \frac{d\bar{z}}{z-t_N} \right\} = -2\pi i \, e^{-2i\alpha} = -2\pi i \, \dot{\bar{t}}^{\,2} ,$$

or, by integration by parts,

$$\lim_{\varepsilon \to 0} \left\{ \iint_{N'} \frac{dxdy}{(z-t_P)^2} - \iint_{N'} \frac{dxdy}{(z-t_N)^2} \right\} = -\pi \, \dot{\bar{t}}^{\,2} . \tag{19}$$

Substitution of (19) and (16) into (11) yields

$$\lim_{\varepsilon \to 0} \left\{ \left(\frac{\partial \psi}{\partial z}\right)^2_{z=z_P} - \left(\frac{\partial \psi}{\partial z}\right)^2_{z=z_N} \right\} = -\frac{\lambda}{4} \, \dot{\bar{t}}^{\,2} . \tag{20}$$

---

* This theorem is an easy consequence of Privaloff's lemma.

(4.2)

We will prove next that $\partial\psi/\partial z$ has measurable limits if γ' is approached from either side non-tangentially:

$$\lim_{z_P \to t} \frac{\partial\psi}{\partial t} \equiv \lim_{\epsilon \to 0} \left(\frac{\partial\psi'}{\partial t}\right)_{z=z_P} = \psi'_P(t)\,;\ \lim_{z_N \to t} \frac{\partial\psi}{\partial t} = \psi'_N(t)$$

for almost all $t \in \gamma'$. Since $\hat{\psi}(\xi + i\eta)$ is a non-decreasing function of ξ for $\xi > 0$, we find

$$-\frac{\pi}{2} < \mathrm{ph}\left(\frac{\partial\hat{\psi}}{\partial\zeta}\right) < \frac{\pi}{2}\,. \tag{21}$$

Let now γ'' denote an arc of γ, which has tangents in its endpoints a, b, and Θ_P a rectifiable arc connecting a and b of finite length inside P. γ'' and Θ_P are chosen so short that in the domain D_P bounded by Θ_P and γ'' for any two points z, z^* in $\bar{D}_P$

$$\left|\mathrm{ph}\, g'(z) - \mathrm{ph}\, g'(z^*)\right| < \pi/2\,. \tag{22}$$

We will consider the point z^* fixed, z variable. Then from (21) and (22)

$$-\pi + \sigma < \mathrm{ph}\left(\frac{\partial\psi}{\partial z}\right) < \pi + \sigma\,, \tag{23}$$

where

$$\sigma = \mathrm{ph}\, g'(z^*)\,.$$

If the function $z = F(Z)$ maps the unit circle R into D_P conformally, then $\frac{\partial\psi}{\partial z} = H(Z)$ is an analytic function of Z in R. Hence by Fatou's theorem we find that there is a measurable function $\tilde{\Phi}(\theta\ \ )$ on the unit circle such that

(4.2)

$$\lim_{Z \to e^{i\theta}} \exp\left(i \log H(Z)\right) = \Phi(\theta)$$

for almost all θ. By the theorems of F. and M. Riesz and Privaloff quoted above, for $t \in \gamma'' \cup \Theta_P$,

(24) $$\lim_{z \to t} \exp\left(i \log \frac{\partial\psi}{\partial z}\right) = \Phi(t)$$

where $\Phi(z)$ is a measurable function defined on ∂D_P. Let Σ denote the subset of ∂D_P on which $\Phi(t) = 0$. Then

$$\lim_{z \to z_o} \mathrm{ph}\left(\frac{\partial\psi}{\partial z}\right) = -\infty \quad \text{if } z_o \in \Sigma,$$

with the exception of a set of measure zero. By (23) however, $\mathrm{ph}(\partial\psi/\partial z)$ has a lower bound hence Σ itself is of zero measure. Thus (24) has the consequence that a measurable function $\psi'_P(t)$, defined on γ'' exists, such that

(25) $$\lim_{z \to t} (\partial\psi/\partial z) = \exp\left(-i \log \Phi(t)\right) = \psi'_P(t)$$

for almost all $t \in \gamma''$, and therefore for almost all $t \in \gamma$.

Consider now domains D_N bounded by γ'' and arcs Θ_N in defined analogously to D_P and Θ_P and inside some circle $|z-z^*| = \rho$. Then we set

(26) $$z_1 = z^* + 2\rho e^{-i\sigma},$$

$$\frac{\partial\psi^*}{\partial z} = \frac{\partial\psi}{\partial z} + \frac{\omega}{2} e^{i\sigma} \operatorname{Re}\left[e^{i\sigma}(z_1 - z)\right].$$

It is easy to check that $\partial\psi^*/\partial z$ is analytic in D_N. With the choice (26) of z_1

$$\operatorname{Re}\left[e^{i\sigma}(z_1 - z)\right] > 0,$$

(4.2)

If σ has the same meaning as there then $\partial\psi/\partial z$ satisfies (23). Therefore also

$$(27) \qquad \sigma - \pi < \mathrm{ph}\,\frac{\partial \psi^*}{\partial z} < \sigma + \pi \,. \qquad (z \in \bar{D}_N)$$

We can prove now that a measurable function $\psi_N^{*\prime}(t)$ defined on γ'' exists such that

$$(28) \qquad \lim_{z \to \measuredangle\, t} \frac{\partial \psi^*}{\partial z} = \psi_N^{*\prime}(t) \,, \qquad (z \in D_N) \,.$$

The proof, based on Eq. (27), is identical to the proof of (25). We define now

$$\psi'_N(t) = \psi_N^{*\prime}(t) - \frac{\omega}{2} e^{i\sigma} \,\mathrm{Re}\left[e^{i\sigma}(z_1 - t)\right].$$

Then from (28)

$$(29) \qquad \lim_{z \to \measuredangle\, t} \frac{\partial \psi}{\partial z} = \psi'_N(t) \qquad (z \in D_N)$$

for almost all t on γ''. (In (29) ψ'_N is a measurable function of the arc length.) Substituting (25) and (29) into (20) we get

$$(30) \qquad \left[\psi'_P(t)\right]^2 - \left[\psi'_N(t)\right]^2 = -(\lambda/4)\,\dot{\bar{t}}^{\,2} \qquad (t \in \gamma)$$

a.e. on γ. For all t for which (25) is valid, we find

$$(31) \qquad \mathrm{ph}\,\psi'_P(t) = \mathrm{ph}\,\psi'_N(t) = -\sigma - \pi/2 \,.$$

Therefore for $z_P - t = \epsilon e^{i(\sigma + \pi/2)}$, ($z_P$ on the normal of γ in t) we find by integration

$$\psi(z_P) = -2\psi_P'(t)(z_P - t) + o(|z_P - t|) \tag{4.3}$$

hence

(32) $$-\left(\frac{\partial\psi}{\partial n}\right)_P = \lim_{\epsilon\to 0}\left|\frac{\psi(z_P)}{z_P - t}\right| = 2\left|\psi_P'(t)\right|$$

and similarly

(33) $$\left(\frac{\partial\psi}{\partial n}\right)_N = \lim_{\epsilon\to 0}\left|\frac{\psi(z_N)}{z_N - t}\right| = 2\left|\psi_N'(t)\right|$$

Combining (30), (31), (32), (33) and the identity $\dot{\bar{z}} = e^{-i\sigma}$ and substitution into (29) yields

$$\left\{\left(\frac{\partial\psi_P}{\partial n}\right)^2 - \left(\frac{\partial\psi_N}{\partial n}\right)^2\right\}_t = \lambda$$

for almost all $t \in \gamma$.

4.3 Boundedness of the eddy region.

We assume that the eddy region N is unbounded and show that this assumption leads to a contradiction.

Because of Eq. (2.6.5) and its connectedness, $\bar{N}$ has to lie in the strip $0 < \eta < \eta_0$, therefore the "length" of $\hat{N}$, that is $\sup\{\mathrm{Re}(\zeta_1 - \zeta_2) : \zeta_1, \zeta_2 \in \hat{N}\}$ is infinite. Since the set $P[\hat{\psi}]$ is connected in the complex topology, the entire real axis must have a neighborhood in $N[\hat{\psi}]$ in the Euclidean topology. The linear measure of the cross section $\sigma_{\xi'}$ of the set $N[\hat{\psi}]$ with $\xi = \xi'$ tends to zero as $\xi' \to \infty$ because $\mathrm{mes}(\sigma_\xi)$ is by the symmetrization property a decreasing function of ξ, hence it has a limit, and this limit is zero for otherwise $T[\psi]$

would become infinite. The area $A_{\xi'}[\hat{\psi}]$ of the part $\xi > \xi'$ of $N[\hat{\psi}]$ also tends to zero because of the finiteness of the area $A[\hat{\psi}]$. We will show that assuming an infinitely long eddy region,

$$(1) \qquad \int_p^q \left(\frac{\partial \hat{\psi}(\zeta, \bar{\zeta})}{\partial \zeta}\right)^2 d\zeta \to -\frac{1}{4}(1-\lambda/\Lambda)(q-p) \; {}^{*}$$

if p, q are real, $p \to \infty$ and $|p-q| < 1$. The integration here is along the real axis. (1) can be written in the form

$$(1') \qquad \int_p^q \left(\frac{\partial \hat{\psi}}{\partial \eta}\right)^2 d\xi \to (1-\lambda/\Lambda)(q-p) \; .$$

$\left(\frac{\partial \hat{\psi}}{\partial \eta}\right)_{\eta=0}$ is an increasing function of $|\xi|$, since for any $h > 0$, $\hat{\psi}(\xi + ih)$ is increasing for a symmetrized $\hat{\psi}$. Therefore (1) implies that

$$(2) \qquad \left(\frac{\partial \hat{\psi}(\zeta, \bar{\zeta})}{\partial \eta}\right)_{\eta=0} \to (1-\lambda/\Lambda)^{\frac{1}{2}} \text{ as } \xi \to \infty,$$

uniformly in any finite interval. On the other hand it will be shown that if the real axis belongs to $\partial\hat{N}$, then

$$(3) \qquad \int_p^q \frac{\partial \hat{\psi}(\zeta, \bar{\zeta})}{\partial \eta} \, d\xi \to 0$$

as $p \to \infty$, contradicting (1').

We start with the proof of (3).

---

* In case of open flows, we set $\Lambda = 1$.

(4.3)

We introduce the following notations:

N_p^q for the segment $p < \xi < q$ of $\hat{N}$,

A_p^q for the area of N_p^q, (if $q = \infty$, it is omitted),

σ_p for the segment $\xi = p$ of $\hat{N}$.

From

$$\nabla^2 \psi = \omega$$

follows

(4) $$\nabla_\zeta^2 \hat{\psi} = \omega |f'(\zeta)|^2$$

hence by Gauss' identity and by $|f'(\zeta)|^2 \geq 1/\Lambda$,

$$-\omega A_p / \Lambda \leq \iint_{N_p} \nabla^2 \psi \, d\xi \, d\eta = \iint_{\partial N_p} \frac{\partial \psi}{\partial n} ds$$

$$\leq - \int_{\sigma_p} \frac{\partial \psi}{\partial \xi} d\eta + \int_p^\infty \left(\frac{\partial \psi}{\partial \eta} \right)_{\eta = 0} d\xi \equiv -I(p) - \int_{\sigma_p} \frac{\partial \psi}{\partial \xi} d\eta .$$

Integrating with respect to ξ we get

(5) $$\omega A_p / \Lambda \geq \int_p^{p+1} I(\xi) d\xi + \int_p^{p+1} d\xi \int_{\sigma_\xi} \frac{\partial \hat{\psi}}{\partial \xi} d\eta .$$

By Schwarz's inequality

(6) $$\left\{ \int_p^{p+1} d\xi \int_{\sigma_\xi} \frac{\partial \psi}{\partial \xi} d\eta \right\}^2 \leq A_p^{p+1} \int_p^{p+1} d\xi \int_{\sigma_\xi} \left(\frac{\partial \hat{\psi}}{\partial \xi} \right)^2 d\eta \leq D\left[\hat{\psi} \,|\, N_p^{p+1} \right] A_p^{p+1} .$$

Hence taking into consideration that I(p) is a decreasing function, we obtain from (5)

$$I_p \leq \frac{\omega}{\Lambda} A_{p-1} + \left(D[\hat{\psi}_-] A_{p-1} \right)^{\frac{1}{2}} \to 0 .$$

(4.3)

It remains to prove (1). We can write the integral equation (4.1.13) in the form

$$4\pi\left(\frac{\partial\hat{\psi}}{\partial\tau}\right)^2 + \pi = -\lambda|f'(\tau)|^2\left[I_1(\tau)+J_1(\tau)\right] + 2\nu|f'(\tau)|^2\left[I_2(\tau)+J_2(\tau)\right]$$

$$-I_3(\tau)-J_3(\tau)-I_4(\tau)-J_4(\tau)+H(\tau),$$

where

$$I_1(\tau) = \iint_{N'} \frac{d\xi d\eta}{(\zeta-\tau)^2},$$

$$I_2(\tau) = \iint_{N'} \frac{\partial\hat{\psi}}{\partial\zeta}\frac{d\xi d\eta}{\zeta-\tau},$$

$$I_3(\tau) = \iint_{N'} \frac{|f'(\zeta)|^2-|f'(\tau)|^2}{(\zeta-\tau)^2}\left[2\nu\hat{\psi}(\zeta,\bar{\zeta})-\lambda\right]d\xi d\eta,$$

$$I_4(\tau) = \iint_{N'} \frac{\dfrac{f''(\zeta)}{f'(\zeta)}}{}\,\frac{d\xi d\eta}{\zeta-\tau},$$

$$H(\tau) = \iint_{N''}\left[2\nu\hat{\psi}(\zeta)-\lambda\right]\left\{\hat{K}(\zeta,\tau)+\overline{\hat{K}(\zeta,\tau)}\right\}d\xi\, d\eta,$$

where

$$N' = \hat{N}_{\tau/2} = \{\zeta : |\xi| > \tau/2,\ \zeta\in\hat{N}\},$$

$$N'' = \hat{N}_0^{\tau/2} = \{\zeta : 0<\xi<\tau/2,\ \zeta\in\hat{N}\}$$

The functions J_1, J_2, J_3, J_4 are obtained from I_1, I_2, I_3, I_4 by replacing $1/(\zeta-\tau)$ by $1/(\bar{\zeta}-\tau)$ in the corresponding formulas.

(4.3)

We first show that $H, I_2, I_3, I_4, J_2, J_3, J_4$ tend to zero as $\tau \to \infty$. $H(\tau)$ clearly tends to zero because in $H(\tau)$ the function $(\zeta-\tau)^{-1}$ is majorized by $2/\tau$, and all cofactors of $(\zeta-\tau)^{-2}, (\zeta-\tau)^{-1}, (\bar{\zeta}-\tau)^{-2}, (\bar{\zeta}-\tau)^{-1}$ in the integrals are uniformly bounded. The integral $I_2(\tau)$ is estimated with the method applied for $J(\tau)$ in section 4.1. Thus by integration by parts

$$\iint_{N'} \frac{\hat{\psi}(\zeta)}{(\zeta-\tau)^2}\, d\xi\, d\eta = -\frac{i}{2}\int_{\partial N'} \frac{\hat{\psi}(\zeta)}{\zeta-\tau}\, d\bar{\zeta} + \iint_{N'} \frac{\partial\hat{\psi}}{\partial\bar{\zeta}} \frac{d\xi\, d\eta}{\zeta-\tau}$$

$$= -\frac{i}{2}\int_{\sigma_{\tau/2}} \frac{\hat{\psi}(\zeta)}{\zeta-\tau}\, d\bar{\zeta} + \iint_{N'} \frac{\partial\hat{\psi}}{\partial\bar{\zeta}} \frac{d\xi\, d\eta}{\zeta-\tau}$$

$$= -\frac{i}{2}\int_{C} \frac{\hat{\psi}(\zeta)}{\zeta-\tau}\, d\bar{\zeta} - \frac{i}{2}\int_{\partial N'} \frac{\bar{\zeta}-\bar{\tau}}{\zeta-\tau} \frac{\partial\hat{\psi}}{\partial\bar{\zeta}}\, d\zeta - \frac{\omega}{4}\iint_{N'} \frac{\bar{\zeta}-\bar{\tau}}{\zeta-\tau} |f'(\zeta)|^2\, d\xi\, d\eta,$$

where $\sigma_{\tau/2}$ is the intersection of the line $\xi = \tau/2$ with N'. In the last evaluation of the integral Eq. (4) was used. The first right hand integral converges to zero since $|\zeta-\tau| > \tau/2$ and $\mathrm{mes}_1(\sigma_{\tau/2}) \to 0$. The seccnd integral has zero limit by (4.2.4). The third is majorized by

$$\iint_{\hat{N}} |f'(\zeta)|^2\, d\xi\, d\eta \leq \frac{1}{\mu^2} A(N_{\tau/2})$$

which also has 0 limit. Therefore $I_2(\tau)$ also tends to zero as $\tau \to \infty$. The terms I_3, I_4 can be estimated together, since

$$(7) \qquad \frac{|f'(\zeta)|^2 - |f'(\tau)|^2}{(\zeta-\tau)^2} = \frac{H(\zeta,\tau)}{\zeta-\tau}$$

where $H(\zeta,\tau)$ is bounded in the whole plane. Observing that $I_4(\tau)$ is also of the form (7), we find sufficient to estimate only integrals of this form. Thus

(4.3)

$$|I_3(\tau)| \le \iint\limits_{N'} \frac{|H(\zeta,\tau)|}{|\zeta-\tau|}\, d\xi d\eta \le C_1 \iint\limits_{N'} \frac{d\xi d\eta}{|\zeta-\tau|} .$$

Let Ω denote the intersection of N' with the circle

$$|\zeta-\tau| < \sqrt{A_{\tau/2}} .$$

We write

$$|I_3(\tau)| \le C_1 \left\{ \iint\limits_{\Omega} \frac{d\xi d\eta}{|\zeta-\tau|} + \iint\limits_{N'-\Omega} \frac{d\xi d\eta}{|\zeta-\tau|} \right\} = C_1(I' + I'') .$$

Here

$$I' \le \int_0^{\sqrt{A_{\tau/2}}} dr \int_0^{2\pi} d\varphi = 2\pi\sqrt{A_{\tau/2}}$$

and

$$I'' \le \iint\limits_{N'-\Omega} \frac{d\xi d\eta}{\sqrt{A_{\tau/2}}} \le \sqrt{A_{\tau/2}} ,$$

hence indeed

(8) $$|I_3(\tau)| \le C_1(2\pi+1)\sqrt{A_{\tau/2}} \to 0, \text{ as } \tau\to\infty,$$

and similarly

$$|I_4(\tau)| \to 0 \text{ as } \tau\to\infty.$$

The same arguments hold for $J_2(\tau), J_3(\tau), J_4(\tau)$ hence these integrals also tend to zero as $\tau\to\infty$. We thus found that

(9) $$4\pi\left(\frac{\partial\hat{\psi}}{\partial\tau}\right)^2 + \pi = \lambda\,|f'(\tau)|^2 \left\{ \iint\limits_{N'} \frac{d\xi d\eta}{(\zeta-\tau)^2} + \iint\limits_{N'} \frac{d\xi d\eta}{(\bar{\zeta}-\tau)^2} \right\} + \vartheta(\tau) ,$$

$$(4.3)$$

where $\theta(\tau)\to 0$ as $\tau\to\infty$. It is known that if Ξ is any path inside $\hat{N}'$ connecting the points $p, q \in \overline{\hat{N}[\psi]}$, then

$$\int_{\Xi} d\tau \iint_{N'} \frac{d\xi d\eta}{(\zeta-\tau)^2} = \iint_{N'} \left(\frac{1}{\zeta-q} - \frac{1}{\zeta-p}\right) d\xi d\eta + \pi(\overline{q-p})$$

(cf. Garabedian-Spencer (1952)). This is even true if p, q are on $\partial\hat{N}'$. We set

$$\int_{p}^{\tau} d\tau \iint_{N'} \frac{d\xi d\eta}{(\zeta-\tau)^2} = G(\tau), \; |f'(\tau)|^2 = F(\tau) .$$

Then by integration by parts

$$\int_p^q F(\tau)G'(\tau)d\tau = [F(q)-F(p)]G(q)$$

$$+ F(p)[G(q)-G(p)] - \int_p^q F'(\tau)G(\tau)d\tau .$$

Here G(q) is uniformly bounded, and since

$$\lim_{p\to\infty} F(p) = \alpha^2, \; \lim_{\tau\to\infty} F'(\tau) = 0 ,$$

we get

$$\lim_{p\to\infty} \int_p^q F(\tau)G'(\tau)d\tau = \lim_{p\to\infty} F(p)[G(q)-G(p)]$$

$$= \alpha^2 \pi(\overline{q-p}) + \lim_{p\to\infty} \iint_{N'} \left(\frac{1}{\zeta-q} - \frac{1}{\zeta-p}\right) d\xi d\eta .$$

(4.4)

The last integral has zero limit (see Eq. (8)). Hence

$$(10)\qquad \lim_{p\to\infty}\int_p^q |f'(\tau)|^2\, d\tau \iint_{N'} \frac{d\xi d\eta}{(\zeta-\tau)^2} = \alpha^2\pi(q-p)\ .$$

Quite similarly

$$\int_p^q d\tau \iint_{N'} \frac{d\xi d\eta}{(\zeta-\tau)^2} = \iint_{N'} \left(\frac{1}{\zeta-q} - \frac{1}{\zeta-p}\right) d\xi d\eta,$$

hence by the same steps

$$(11)\qquad \lim_{p\to\infty}\int_p^q |f'(\tau)|^2\, d\tau \iint_{N'} \frac{d\xi d\eta}{(\zeta-\tau)^2} = 0\ .$$

Substitution of (10) and (11) into (9) yields (1).

4.4 Lemma. Suppose that $\psi_1(z)$, $\psi_2(z)$ are solutions of IV in the domains Δ_1, Δ_2 respectively, for the same values ω, λ. If there is an open nonempty domain $G \subset \Delta_1 \cap \Delta_2$, such that $\psi_1(z)=\psi_2(z)$ on ∂G, then (a) $P[\psi_1] = P[\psi_2] = S$ and $\psi_1(z) = \psi_2(z)$ in S; (b) Either one of the sets $N[\psi_1]$, $N[\psi_2]$ is empty or $N[\psi_1] = N[\psi_2] = R$ and $\psi_1(z) = \psi_2(z)$ in R.

Proof. We introduce the functions

$$\mu_1(z) = \begin{cases} \psi_2(z) \text{ in } G \\ \psi_1(z) \text{ in } \Delta_1 - G\ , \end{cases}$$

$$\mu_2(z) = \begin{cases} \psi_1(z) \text{ in } G \\ \psi_2(z) \text{ in } \Delta_2 - G\ . \end{cases}$$

(4.4)

Clearly both functions are admissible for IV in their domains of definition. Furthermore, it is easy to verify the identities

$$T[\mu_1] + T[\mu_2] = T[\psi_1] + T[\psi_2],$$

$$L[(\mu_1)_-] + L[(\mu_2)_-] = L[\psi_1)_-] + L[(\psi_2)_-],$$

$$A[\mu_1] + A[\mu_2] = A[\psi_1] + A[\psi_2].$$

Consequently

$$W[\mu_1] + W[\mu_2] = W[\psi_1] + W[\psi_2].$$

This implies that

$$W[\mu_1] \leq W[\psi_1] \text{ or } W[\mu_2] \leq W[\psi_2].$$

Suppose that e.g., the first inequality holds. If μ_1 does not belong to $\mathcal{M}$, then the solution $\nu(z) \in \mathcal{J}\mathcal{M}$ of the restricted minimum-problem in Δ_1 defined by $O[\mu_1]$ satisfies

$$W[\nu] < W[\mu_1] \leq W[\psi_1]$$

which is impossible. Therefore $\mu_1(z) \in \mathcal{J}_{\Delta_1}\mathcal{M}$.

Suppose first that $\psi_1(z) \leq 0$ is not everywhere true on ∂G. Then we define the sets

$$P_1^* = P[\psi_1] \cap G, \quad P_1^{**} = P[\psi_1] \cap (\Delta_1 - G),$$

$$P_2^* = P[\psi_2] \cap G, \quad P_2^{**} = P[\psi_2] \cap (\Delta_2 - G),$$

and

$$p = \partial P_i^* \cap \partial P_i^{**} \qquad (i = 1, 2)$$

(4.4)

where

$$p_i \subset \partial G ,$$

and p_1 is not empty. The function $u_1(z)$ is harmonic in P_1^*, and

$$u_1(z) - \psi_1(z) = 0$$

there. If p_1 is not empty, then $P_1^{**} \cap P_2^*$ is not empty because of continuity. Thus by analytic continuation

(1) $$u_1(z) = \psi_1(z) \text{ in } P_1^{**} \cap P_2^{**}$$

But outside G $u_1(z) = \psi_2(z)$ by definition, hence

(2) $$u_1(z) = \psi_2(z) \quad \text{in} \quad S^{**} = P_1^{**} \cap P_2^{**}$$

(S^{**} is not empty because of continuity). By analytic continuation into $P_1^* \cap P_2^*$ it can be shown that (2) is valid in the whole of $S = P[\psi_1] \cap P[\psi_2]$. Hence in particular

$$\psi_1(z) = \psi_2(z) = 0 \quad \text{(open flows)}$$

$$\psi_1(z) = \psi_2(z) = \begin{cases} 0 \\ \pi \end{cases} \text{(channel flows)}$$

on the boundary of S. Since both $P[\psi_1]$ and $P[\psi_2]$ are simply connected,

$$P[\psi_1] = P[\psi_2] = S$$

and

$$\psi_1(z) = \psi_2(z) \text{ in } S.$$

Let us now assume that everywhere on ∂G

$$\psi_1(z) = \psi_2(z) \leq 0 .$$

(4.4)

Then

$$\partial G \subset \overline{N[\psi_1]} \cap \overline{N[\psi_2]} .$$

This implies $G \subset N[\psi_1] \cap N[\psi_2]$, because both right hand sets consist of disjoint simply connected open domains. Thus the function

$$\psi_1(z) - \psi_2(z)$$

is harmonic in G, continuous and vanishing on ∂G, hence

(3) $$\psi_1(z) = \psi_2(z)$$

in G. If R is the union of the open sets where (3) is valid and $\psi_i(z) < 0$, then $\psi_i(z) = 0$ on ∂R. The set $\gamma^* = \partial R \cap \gamma$ cannot be empty since no subset of β is a closed curve, thus cannot form alone a boundary. γ^* is part of the boundary of $S = P[\psi_1] \cap P[\psi_2]$, and for almost all $t \in \gamma^*$ and for $z \in R$

(4) $$\lim_{z \to > t} \frac{\partial \psi_1(z)}{\partial t} = \lim_{z \to > t} \frac{\partial \psi_2(z)}{\partial t} .$$

By the matching condition (4.2.30) the same limit relation holds for $z \in S$, for almost all $t \in \gamma^*$. The function

$$u(z) = \exp\left\{ i \left(\frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_2}{\partial z} \right) \right\} - 1$$

is analytic in P and bounded because of the symmetrization property of $\psi_1(z)$ and $\psi_2(z)$, and as just shown has 0 as non-tangential limit on a subset of ∂S of positive measure. By a theorem of F. and M. Riesz (1916) follows that u(z) vanishes identically in S, or $\psi_1(z) = \psi_2(z)$ there. This also implies that

$$P[\psi_1] = P[\psi_2] = S .$$

(4.4)

(b) First we observe that if $N[\psi_i]$ consists of more than one disjoint open component, then all but one of these components have their boundaries made up of sections of ∂P exclusively. This follows from the symmetrization property of ψ_i. These components are therefore completely determined by P, and (except possibly their ordering) are the same for ψ_1 and ψ_2. The functions $\psi_1(z)$, $\psi_2(z)$ are in turn determined in these components because of the uniqueness of the solution of the restricted inner minimum-problem. Thus we only need to concern ourselves with the components N_1', N_2', which are bounded; in addition to sections of ∂P, by arcs of the bounding streamlines β_1, β_2 respectively.

If, $N[\psi_1]$ and $N[\psi_2]$ are not empty, neither are N_1' and N_2', for otherwise $\partial N_i \cap \beta_i$ would consist of a single point, contradictory to the result 3.6. By the symmetrization property $\partial P \cap \partial N_1' \cap \partial N_2'$ contains an arc γ^{**} of non-zero length, ans since $\psi_1 = \psi_2$ in P, (4) is valid with $z \in P$ and for almost all $t \in \psi^{**}$. From this follows that $N_1' = N_2' = N'$ and $\psi_1 = \psi_2$ in N' by an argument entirely similar to the one applied after Eq. (4).

Corollary 1. If IV for specified ω, λ and Δ has two different solutions ψ_1, ψ_2, then $\psi_1 \geq \psi_2$ in Δ, or vice versa.

Suppose the statement is not true. Then let S denote the set on which $\psi_1 < \psi_2$; S and $\Delta - \bar{S}$ are not empty by the assumption. Then $\psi_1 = \psi_2$ on ∂S. Therefore by the lemma $\psi_1 = \psi_2$ in Δ.

Corollary 2. If $\Delta = E$, then the only solution of IV is the trivial. Otherwise with the solution $\psi(z)$ all functions $\psi(z+c)$, c real, would be solutions. Thus $\psi(z+c) = \psi(z)$ for all c, which is only possible for $\psi = y$.

Thus at least one case is found when the only solution of IV is the trivial one. However, for most other domains non-trivial solutions exist for some values of ω, as was shown in Sections 1.5, 2.9.

4.5 Theorem. If $\psi(z)$ is a solution of IV, then $\psi(z)$ has only one relative minimum (on the imaginary axis, of course) and therefore the set $N[\psi]$ is connected*.

Proof. Suppose $\psi(z)$ has relative minima in ip, iq, and a maximum in ir, $p<r<q$. Then for sufficiently small $h>0$,

$$(1) \qquad \begin{cases} \psi(ip-ih) > \psi(ip) \ , \\ \psi(ip) \quad < \psi(ip+ih) \ , \\ \psi(ir) \quad > \psi(ir+ih) \ , \end{cases}$$

$$(2) \qquad h<p, \ h<q-r \ .$$

Suppose that S_h is the set in which

$$\psi(z+ih)<\psi(z) \ .$$

S_h is open, therefore it can be represented in the form

$$\hat{S}_h = \sigma_1 \cup \sigma_2 \cup \dots$$

---

* The reader may be reminded that the results of Section 3.1 only assert the connectedness of the closure of N.

where σ_k are connected open sets which are pairwise disjoint. Since ψ is symmetrized, if some component σ_k contains a point $\xi_o + i\eta_o$ it contains the point $i\eta_o$ as well. Therefore from the relationships (1) follows that there is a component, say σ_1, in the strip

$$g(i(p-h)) < \eta < g(i(r+h))$$

On the boundary of σ_1

(3) $$\psi(z+ih) = \psi(z) .$$

This implies by the previous lemma that (3) is valid everywhere in $P[\psi]$, which is impossible.

4.6 <u>The domain N of the free-eddy solution* of problem III is connected.</u>

<u>Proof</u>. If N is not connected, then there is a $y_o > 0$ such that

$$N_1 = \{z\colon y > y_o,\ z \in N\}$$

and

$$N_2 = N - N_1 = \{z\colon y < y_o,\ z \in N\}$$

are disjoint. The intersection of the set N with any line $y = \text{const.}$ is an interval symmetric to the imaginary axis. Given any $\epsilon > 0$, one can choose the positive numbers h_1, h_2 such that the intersections with the lines $y = y_o - h_1 = b$, $y = y_o + h_2 = b + h$ are of equal length $2a$, and that the segment of N between these lines is inside a circle $|z - iy_o| < \epsilon$.

---

* i.e., the solution for the domains $\Delta = E, E$ either the strip $0 < \eta < \pi$ or the halfplane $\eta > 0$.

Consider the sets

$$N_1^* = \{z: y > b+h,\ z \in N\}$$

$$N_2^* = \{z: y < b,\ z \in N\},$$

$$N_2^{**} = \{z: z+h \in N_2^*\};$$

$$N^* = N_1^* \cup N_2^*,\quad N^{**} = \operatorname{Int}(\overline{N_1^* \cup N_2^{**}}).$$

Thus the set N^* arises from N by slicing off the tips of N_1 and N_2 lying in the region $b<y<b+h$, and N^{**} is obtained from N^* by shifting the upper disjoint part N_2^* downwards until it touches the lower part N_1^* and forming of the two sets a single open set connected on the line $y=b$. The potential flow around N^* which satisfies the appropriate boundary conditions at ∞, β (and on β') is a function $\psi^*(z)$ and the one around N^{**} is a function $\psi^{**}(z)$. It is clear that

$$\text{(1)}\qquad T[\psi_+^*] \le T[\psi_+]$$

because the function

$$\psi_+^* = \begin{cases} \psi_+(z) & \text{in } P[\psi] \\ 0 & \text{in } P[\psi^*] - P[\psi] \end{cases}$$

is a competing function for the outer minimum-problem for the domain N^*. If any translations of the upper part N_2^* of N^* are allowed, then $T[\psi^*]$ will increase if N_2^* is translated horizontally. In fact, such translation happens to be for $\Delta = E$ (free eddy) the inverse operation of symmetrization, and the latter is known to reduce the functional T. Let $\psi_c(z)$ denote the stream function obtained by translation of N_2^* by the complex vector c. Then, as just shown, ψ_c is a concave function of c for real c, and therefore by theorem 3.1, concave for imaginary c. On the other hand,

$$T[\psi_c] \to +\infty \text{ as } c \to +i\infty .$$

(4.6)

Therefore there is a constant K for any $C>0$, such that for $c = ic_1$, (c_1 real)

(2) $$\frac{d}{dc_1} T[\psi_c] \geq K > 0$$

if $c_1 < C$, whenever c_1 is such that the translated $\overline{N_2^*}$ does not intersect N_1^*, i.e., for $C > c_1 > -h$.

From (2) follows that

$$T[\psi_+^*] \geq T[\psi_+^{**}] + Kh,$$

and in combination with (1),

(3) $$T[\psi_+] \geq T[\psi_+^{**}] + Kh .$$

The set N_1' obtained from N_1 by a downward shift $-ih$ intersects N_2^* in the set N_o, which is "symmetrized" to the imaginary axis. N_o is bounded by the curves

$$\sigma_1 = \{z: z \in \partial N_2, \ y > b\},$$

$$\sigma_2 = \{z: z \in \partial N_1', \ y \leq b\}.$$

Then

$$\psi(z+ih) - \psi(z) < 0 \quad \text{on } \sigma_1 ,$$

$$\psi(z+ih) - \psi(z) > 0 \quad \text{on } \sigma_2 .$$

Therefore there is a Jordan-curve θ connecting the points $ib \pm a$ inside N_o, such that $\psi(z+ih) = \psi(z)$ along θ. We define now a new admissible function $\Psi(z)$ of the minimum problem III as follows. The curve θ with the portion of ∂N_2 under $y = b$ bound a domain M_2, and with the portion of $\partial N_1'$ over $y = b$ the domain M_1. We introduce

(4.6)

$$\Psi(z) = \begin{cases} \psi^{**}(z) & \text{in } P^{**} = \Delta - \overline{N^{**}} , \\ \psi(z) & \text{in } M_2 , \\ \psi(z+ih) & \text{in } M_1 . \end{cases}$$

$\Psi(z)$ is clearly continuous, and therefore by its definition admissible for the problem III or IV. Further

$$(4) \qquad T[\Psi_+] = T[\psi^{**}] \leq T[\psi_+] - Kh,$$

$$(5) \qquad D[\Psi_-] = D[\psi | M_2] + D[\psi^{**} | M_1] < D[\psi_-] ,$$

$$(6) \qquad A[\Psi] = A(M_1) + A(M_2) = A[\psi] - A(N_o) \geq A[\psi] - 2\,sh,$$

$$(7) \qquad L[\Psi_-] = L[\psi_-] - L[\psi_- | N_2 - M_2] - L[\psi_-(z+ih) | N_1^{**} - M_1] \geq L[\psi_-] - 4\,sh\mu$$

where

$$\mu = \max_N |\psi(z)|$$

and

$$s = \sup \left\{ x : \psi(x+iy) < 0,\ b < y < b+h \right\}$$

or, in other words: $2s$ is the maximal width of N_o.

Assuming that $L[\psi] = m$, (which can be achieved by normalization), from (4), (5), (6) and (7) we find easily

$$V'[\Psi] < V'[\psi] - \left(K - 2\lambda s - 4T[\psi]\frac{\mu s}{m}\right) h + O(s^2 h^2) .$$

Since $s \to 0$ as $h \to 0$, we find that for sufficiently small h,

$$V'[\Psi] < V'[\psi] ,$$

a contradiction.

SUMMARY OF RESULTS AND CONCLUDING REMARKS

This work gives a partial answer to Batchelor's problem outlined in the introduction. The existence of solutions is shown in flows in regions bounded by a single or two streamlines, and characterized by mapping functions $f(\zeta)$ described in Section 1.1. Considering all functions u admissible for which the functionals involved are finite, the minimum-problems

(III) $$T[u] - \lambda A[u] = \min$$

(side condition: $L[u] = m$) and

(IV) $$T[u] - \lambda A[u] - \omega L[u] = \min$$

(no side condition) were formulated. They were found "well-posed" in the sense of Section 1.5 if $\lambda < \Lambda = 1/f'(\infty)^2$ and in case of problem IV, channel-flows also

$$\frac{1}{12\Lambda}\left(\frac{\omega}{\pi}\right)^2 < \phi\left(\frac{\lambda}{\Lambda}\right)$$

where

$$\phi(X) = \inf_{\frac{1}{2}<\rho<\frac{2}{3}} \left\{\frac{1}{\rho^2}\left(\frac{1}{1-\rho} - X\right)\right\} .$$

These bounds cannot be improved. Problem IV is never well-posed and has no solution for open flows.

It was found that well-posed problems III and IV have solutions $\psi(z)$ which are admissible. These solutions satisfy

$$\nabla^2\psi = \omega S(\psi) ,$$

where $S(\psi) = 1$ if $\psi < 0$ and $= 0$ if $\psi > 0$; ψ is continuous in $\bar{\Delta}$, and assumes the boundary value(s) 0, (π) on the bounding streamlines. Further

$$\psi = \Lambda^{\frac{1}{2}} y + O(1) \qquad (|z| \to \infty) .$$

The function $\psi(z) = \hat{\psi}(\xi + i\eta)$ is even in ξ and an increasing function of $|\xi|$ ("symmetrized"). The set N is bounded. The sets $\bar{N} = \{\psi(z) \leq 0\}$ and $P = \{z : \psi(z) > 0\}$ are simply connected, and in case of problem IV or in case of the "free eddy" problem even the open set N is simply connected. Further N and P have no "internal" boundary points : $\partial P = \partial\bar{P}$, $\partial N = \partial\bar{N}$. The function $|\nabla\psi|$ has a uniform positive lower bound in P, and if the boundary β' is empty (open flows) or straight, then this lower bound is $\lambda^{\frac{1}{2}}$. Consequently the boundary

$$\gamma = \partial P \cap \partial N$$

is a rectifiable curve of finite length. $\partial\psi/\partial z$ has non-tangential limits $\psi'_P(t)$, $\psi'_N(t)$ for almost all $t \in \gamma$, if t is approached from either side of γ, and

$$\left[\psi'_P(t)\right]^2 - \left[\psi'_P(t)\right]^2 = -(\lambda/4)\,\dot{\bar{t}}^2 .$$

This implies the weaker result :

$$\left(\frac{\partial\psi}{\partial n}\right)^2_P - \left(\frac{\partial\psi}{\partial n}\right)^2_N = \lambda$$

almost everywhere on γ .

The solution of IV is not necessarily non-trivial, i.e., the set N may be empty. Such is the case if Δ has straight boundaries. The solution of IV is non-trivial if $\lambda > 1/|f'(o)|^2$. Uniqueness of the solution of Batchelor's problem was not proved and in some cases the solution is demonstrably not unique. Nevertheless the solutions satisfy another criterion of being physically well-posed ; the set of solutions for given Δ, λ and m or ω depends continuously on these.

The present study falls short both in the theoretical and the practical sense from the desired goal. From the theoretical point of view, it does not answer some very important questions relative to the smoothness of the boundary. It seems very plausible especially in view of the results of Garabedian, Spencer, Lewy, Schiffer on the analiticity of the free boundary in cavitation flows, that the boundary γ is smooth or even can be described by infinitely differentiable functions. No such results could be proved for Batchelor-flows. It is also plausible that at the point of separation the curve γ has a tangent which then has to be the tangent of the curve β in that point. Attempts to prove this were also mostly unsuccessful. Also, no explicit solutions in particular cases are in sight, such as the ones found for cavity flows.

The more general case of domains not permitting symmetrization is interesting because it includes a model of the wake formation in flows behind bounded symmetric obstacles. Preliminary investigations indicated that the present analysis could be broadened to include more general domains. However then all arguments based on the powerful tool of symmetrization have to be replaced, and some results will be lost or weakened in the process. In particular, the connectedness and boundedness of the eddy domain cannot be guaranteed, only that the components of the eddy region are adjacent to the domain boundary. This is quite natural, for if e. g. β consists of widely separated indentations connedted by intervals of the real axis, it is quite plausible that the wake will not be connected, nor bounded if the indentations extend to infinity. The matching condition can be proved in a slightly weaker form.

From the practical point of view, no attempt has been made to connect this analysis with the theory of the boundary layers. Anyway no boundary layer analysis is possible in the case of infinitely long boundary lines. In the case of "free eddies" this objection can be eliminated since reflection to the real axis may produce a flow region without boundaries or with finite boundaries. In the case of flows around finite bodies not included in the present study, the latter may lend enough plausibility to the existence of the flow to encourage an attempt for the numerical determination of such flows based on the minimum-principles discussed. However, any numerical study is made rather difficult since the two unknown parameters (λ and ω or m) have to be determined from the matching of outer, inner, and boundary layer flows near the separation and reattachment points. In the case of flows in halfplanes (free eddies) the set of solutions reduces essentially to a one-parameter family by similarity considerations. Since arbitrary sections of the real axis wetted by the eddy domain can be replaced by flat plates, it seems possible that any choice of the parameters λ, m determines a limit case of viscous flows.

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Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

Polytechnic Institute of Brooklyn
Dept. of Aerospace Engrg. & Applied Mechanics
Route 110, Farmingdale, New York 11735

2a. REPORT SECURITY CLASSIFICATION

Unclassified

2b. GROUP

3. REPORT TITLE

ON TWO DIMENSIONAL INCOMPRESSIBLE STEADY STATE FLOWS WITH SEPARATION

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Research Report

5. AUTHOR(S) (First name, middle initial, last name)

Alexander Pal

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
October 1968	131	24

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
NONR 839(38)	PIBAL Report No. 68-29
b. PROJECT NO.	
c. ARPA Order No. 529	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. DISTRIBUTION STATEMENT

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Office of Naval Research Department of the Navy Washington, D. C.

13. ABSTRACT

The subjects of this study are two-dimensional incompressible steady state flows which have constant vorticity (ω) in a domain N boundary by a closed streamline and are irrotational in the complementary part P of the flow domain, and such that the stream function $\psi(z)$ $(z=x+iy)$ satisfies on the boundary $\gamma=\partial P\ \partial N$ Bernoully's law

$$(\partial\psi/\partial n)_P^2 - (\partial\psi/\partial n)_N^2 =$$

According to G. K. Batchelor such flows (below called "Batchelor-flows") may be models of laminar flows exhibiting separation phenomena in case of high Reynolds numbers. A two-parameter family of Batchelor-type flows in certain symmetric plane domains bounded by one streamline (open flows) or two streamlines (channel flow) are obtained as solutions of minimum problems by direct methods of variational calculus.

The minimum problems in question are of the type of, or are formally equivalent to

$$T[\psi]-\lambda A[\psi]-\omega L[\psi] = \min$$

where T is a modified Dirichlet-integral analogous to the virtual mass, $A[\psi]$ the area of the region where $\psi<0$, $L[\psi]$ the angular momentum in this region.

Some of the proven properties of the solutions: The region N is defined by the conditions $\psi(z)<0$ and is bounded. The flow is asymptotically uniform at large distances. The sets N ∂N and P are simply connected and ∂N contains a [illegible] arc of $\partial\Delta$. Of particular interest are separated flows in domains with one or two straight bounding streamlines.

DD FORM 1 NOV 65 1473

Unclassified

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Variational calculus, (direct methods of)						
Flows with separation						
Wakes						
Laminar flows at high Reynolds numbers						
Virtual mass						